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Coherent states over Grassmann manifolds and the WKB-exactness in path integral

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Abstract

$\mathrm{U}(N)$ coherent states over Grassmann manifold, $G_{N,n} \simeq \mathrm{U}(N)/(\mathrm{U}(n) \times \mathrm{U}(N-n))$, are formulated to be able to argue the WKB-exactness, so called the localization of Duistermaat-Heckman, in the path integral representation of a character formula. The exponent in the path integral formula is proportional to an integer k that labels the $\mathrm{U}(N)$ representation. Thus when $k \to \infty$ a usual semiclassical approximation, by regarding $k \sim 1/\hbar$, can be performed yielding to a desired conclusion. The mechanism of the localization is uncovered by means of a view from an extended space realized by the Schwinger boson technique.

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1 Introduction

In any physical situation it is often difficult to find an exact response, therefore, some approximation method must be employed. Apart from the well-known perturbation theories, the Wentzel-Kramers-Brillouin(WKB) approximation, known as the $\hbar$-expansion, seems most suitable to the path integral formalism; since the exponent in the path integral representation is usually given by a quantity divided by Planck's constant $\hbar$. These approximation methods can be straightforwardly performed without specifying a path measure rigorously, which apparently has been a main reason that path integral plays a major role in modern physics. On the other hand, due to this facileness there always accompanies some skepticism, such as the problem of operator ordering[1] since only c-numbers appear, or a vague relationship of change of variables to the canonical operator formalism[2]. What we have learned through various efforts is that path integral can produce reliable as well as consistent results under the time slicing method. A simple example can be seen as follows: take a bosonic oscillator, defined by a Hamiltonian $H \equiv \omega a^\dagger a$ with the algebra $[a, a^\dagger] = \mathbf{I},\ [\mathbf{I}, a] = [\mathbf{I}, a^\dagger] = 0$, and calculate the quantity $\mathrm{Tr} e^{-iHT}$. First write the exponential operator such that

$$\mathrm{Tr} e^{-iHT} = \lim_{M\to\infty} \mathrm{Tr}\left(\mathbf{I} - i\Delta t H\right)^M ; \quad \Delta t \equiv \frac{T}{M} \quad , \tag{1.1}$$

which is the starting point of the time slicing method. With the aid of canonical coherent states[3], (1.1) becomes

$$\begin{aligned} \mathrm{Tr} e^{-i\omega a^\dagger a T} &= \lim_{M\to\infty} \int_{\mathrm{PBC}} \prod_{j=1}^{M} \frac{dz(j) d\bar{z}(j)}{\pi} \\ &\times \exp\left[-\sum_{k=1}^{M} \left\{ \bar{z}(k)(z(k) - z(k-1)) + i\omega\Delta t \bar{z}(k) z(k-1) \right\} \right], \end{aligned} \tag{1.2}$$

where "PBC" denotes $z(0) = z(M)$ and $dz d\bar{z} \equiv d\mathrm{Re}(z) d\mathrm{Im}(z)$. By taking a formal limit, $M \to \infty$, (1.2) yields to the continuum representation,

$$\mathrm{Tr} e^{-i\omega a^\dagger a T} \to \int_{\mathrm{PBC}} \prod_{0\le t\le T} dz d\bar{z} \exp\left\{ -\int_0^T dt \left(\bar{z}\dot{z} + i\omega \bar{z} z \right) \right\} \quad , \tag{1.3}$$

where "PBC" reads $z(T) = z(0)$ in this case. In spite of the formal limit, we still could endow a meaning with the functional measure by means of the functional determinant:

$$(1.3) \equiv \det\left(\frac{d}{dt} + i\omega \right)^{-1} = \frac{1}{2i\sin(\omega T/2)} \quad , \tag{1.4}$$

with the aid of the ζ-function regularization. The result does not, however, match to the correct one given by (1.2)

$$\frac{1}{2i\sin(\omega T/2)} \neq \frac{1}{1 - e^{-i\omega T}} = \frac{e^{i\omega T/2}}{2i\sin(\omega T/2)} \quad . \tag{1.5}$$

Therefore we must pay the price whenever we have adopted the continuum path integral representation which would apparently be suitable for a geometrical treatment.

In some situation an approximation scheme happens to lead to an exact answer. The harmonic oscillator is WKB-exact, because of the integration being Gaussian. (The cross section of the Coulomb interaction is another well-known example[4], which furthermore reveals that the Born approximation yields the exact result.) In recent years, however, a new possibility of finding the WKB exactness has been opened up[5, 6, 7, 8, 9, 10] being inspired by the Duistermaat-Heckman's (D-H) theorem[11, 12, 13]. A key word to understand these new classes of the WKB exactness would be '*localization*' [14, 15, 16], commonly understood in terms of equivariant cohomology[17].

Inspired by these facts, we have established the WKB exactness of path integral obtained through the generalized coherent states[18] in cases of $\boldsymbol{C}P^1$ [9] and $\boldsymbol{C}P^N$[10] as well as their noncompact counter parts in the foregoing papers. Even if other representation is adopted[19] for the $\boldsymbol{C}P^1$ case to give the Nielsen-Rohrlich form[20], the same localization has been clarified[21]. As a natural generalization in this paper, we try to understand the same phenomena in the case of the Grassmann manifold, $G_{N,n} \simeq \mathrm{U}(N)/(\mathrm{U}(n)\times\mathrm{U}(N-n))$. To this end, we need to build up coherent states of $\mathrm{U}(N)$ over $G_{N,n}$.

The plan of the paper is as follows. An interpretation of the D-H theorem, stated in terms of finite dimensional integrations, is presented in section 2, since the WKB exactness is sometimes declared as a generalization to the infinite dimensional case. In order to formulate quantity as path integral, there need to construct coherent states over Grassmann manifolds: we develop two ways. One is the algebraic method according to the Perelomov's prescription[18] and the other is that in terms of the Schwinger boson[22] as well as the canonical coherent state[3]. These are the contents of section 3.1 and 3.2, respectively. The path integral representation of the character formula is then given in section 4.1 and the WKB approximation is performed in section 4.2. The mechanism of the WKB exactness is clarified in section 4.3 by making use of the Schwinger boson. The final section is devoted to related topics and remarks. In Appendix A, proofs for theorems, utilized in the section 3.1, are given. Finally in Appendix B a discussion on the WKB exactness for the Feynman kernel is given as a supplement; since which is considered to be more general than the discussion in the text.

2 The D-H formula on Grassmann manifolds

In this section, we explain the validity of the D-H formula on $G_{N,n}$ to make a preparation for later discussions.

2.1 Classical mechanics on $G_{N,n}$

Let $\boldsymbol{C}^n$ be the n-dimensional complex vector space. We denote the space of $m\times n$ matrices over $\boldsymbol{C}$ by $M(m,n;\boldsymbol{C})$ and abbreviate $M(n,n;\boldsymbol{C})$ by $M(n;\boldsymbol{C})$.

We identify $G_{N,n} \simeq \mathrm{U}(N)/(\mathrm{U}(n)\times\mathrm{U}(N-n))$ as a phase space, assuming $N\geq 2n$ for brevity's sake, to write

$$G_{N,n} = \{P \in M(N;\boldsymbol{C}) | P^2 = P,\ P^\dagger = P \ \text{ and } \ \mathrm{tr}\, P = n\}\ . \tag{2.1}$$

P can be parameterized in terms of $\xi \in M(N-n,n;\boldsymbol{C})$ such that

$$P = \begin{pmatrix} \dfrac{1}{1_n+\xi^\dagger\xi} & \dfrac{1}{1_n+\xi^\dagger\xi}\xi^\dagger \\ \xi\dfrac{1}{1_n+\xi^\dagger\xi} & \xi\dfrac{1}{1_n+\xi^\dagger\xi}\xi^\dagger \end{pmatrix} = U(\xi)P_{1,\ldots,n}U^\dagger(\xi)\ , \tag{2.2}$$

where

$$P_{1,\ldots,n} \equiv \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix}\ , \tag{2.3}$$

and

$$U(\xi) \equiv \begin{pmatrix} \dfrac{1}{\sqrt{1_n+\xi^\dagger\xi}} & -\dfrac{1}{\sqrt{1_n+\xi^\dagger\xi}}\xi^\dagger \\ \xi\dfrac{1}{\sqrt{1_n+\xi^\dagger\xi}} & \dfrac{1}{\sqrt{1_{N-n}+\xi\xi^\dagger}} \end{pmatrix}\ . \tag{2.4}$$

The parameterization (2.2) cannot cover the whole phase space: indeed there exist other parameterizations such as

$$(P_{\mu_1,\ldots,\mu_n})_{\rho\lambda} = \sum_{a=1}^{n} \delta_{\rho,\mu_a}\delta_{\lambda,\mu_a}, \quad 1\leq \mu_a \leq N, \tag{2.5}$$

which tells us that there need $\binom{N}{n}$ kinds of local parameterization. (Throughout the paper we use a convention for indices: $1\leq\mu,\nu\leq N;\ 1\leq a,b\leq n;\ n+1\leq i,j\leq N$.) In order to obtain an appropriate parameterization in the neighborhood of $P_{\mu_1,\ldots,\mu_n}$, we can utilize a unitary transformation $U(\mu_1,\ldots,\mu_n|1,\ldots,n)$ satisfying

$$U(\mu_1,\ldots,\mu_n|1,\ldots,n)P_{1,\ldots,n}U^\dagger(\mu_1,\ldots,\mu_n|1,\ldots,n) = P_{\mu_1,\ldots,\mu_n}\ . \tag{2.6}$$

The symplectic structure on $G_{N,n}$ is defined through the symplectic 2-form

$$\omega = i\,\mathrm{tr}(PdP\wedge dP)\ , \tag{2.7}$$

whose explicit form, under the above parameterization, is found as

$$\omega = i\,\mathrm{tr}\{(1_n+\xi^\dagger\xi)^{-1}d\xi^\dagger\wedge(1_{N-n}+\xi\xi^\dagger)^{-1}d\xi\} \tag{2.8}$$

yielding to the $\mathrm{U}(N)$ invariant measure on $G_{N,n}$,

$$\begin{aligned}
& \det\left[(1_{N-n}+\xi\xi^\dagger)^{-1}\otimes\{(1_n+\xi^\dagger\xi)^{-1}\}^{\mathcal{T}}\right](d\xi d\bar{\xi})^{n(N-n)} \\
= & \frac{1}{\{\det(1_n+\xi^\dagger\xi)\}^N}(d\xi d\bar{\xi})^{n(N-n)} \ , \qquad (2.9)
\end{aligned}$$

where the superscript $\mathcal{T}$ denotes the transpose of a matrix and an abbreviation

$$(dzd\bar{z})^{mn} \equiv \prod_{\substack{1\le i\le m\\ 1\le a\le n}} d\mathrm{Re}(z_{ia})d\mathrm{Im}(z_{ia}) \ , \qquad (2.10)$$

has been employed. Our convention of tensor product is

$$A\otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots \\ a_{21}B & a_{22}B & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \quad \text{for } A=(a_{ij}),\ B=(b_{ij}) \ .$$

Dynamical variables are defined through a linear mapping,

$$X\in \mathrm{H}(N)\mapsto F_X=\mathrm{tr}(PX)\in \boldsymbol{R}, \qquad (2.11)$$

where $\mathrm{H}(N)$ is the space of Hermitian matrices:

$$\mathrm{H}(m)=\{X|X\in M(m;\boldsymbol{C}),\ X^\dagger=X\} \ . \qquad (2.12)$$

The Poisson bracket is given, with the aid of (2.7), by

$$\{F_X,F_Y\}_{\mathrm{P.B.}}=\omega^{-1}(V_X,V_Y)=F_{-i[X,Y]} \ , \qquad (2.13)$$

with $[X,Y]=XY-YX$, where V_X is a vector field on $G_{N,n}$ associated with F_X:

$$V_X=\sum_{i,a}\left(\frac{\partial F_X}{\partial\xi_{ia}}\frac{\partial}{\partial\xi_{ia}}+\frac{\partial F_X}{\partial\bar{\xi}_{ia}}\frac{\partial}{\partial\bar{\xi}_{ia}}\right) \ .$$

Note that the Poisson bracket (2.13) generates the $u(N)$ algebra.

An explicit form of the Hamiltonian function for a Hermitian matrix X,

$$X=\begin{pmatrix} A & B \\ B^\dagger & D\end{pmatrix},\ A\in\mathrm{H}(n),\ B\in M(n,N-n;\boldsymbol{C}),\ D\in\mathrm{H}(N-n), \qquad (2.14)$$

is read as

$$\begin{aligned}
F_X &= \mathrm{tr}\{(1_n+\xi^\dagger\xi)^{-1}\Phi_X\} \ , \\
\Phi_X &= A+B\xi+\xi^\dagger B^\dagger+\xi^\dagger D\xi \ . \qquad (2.15)
\end{aligned}$$

Introduce a 1-form θ_κ such that

$$d\theta_\kappa = \omega, \quad \theta_\kappa = i\,\mathrm{tr}[\{\kappa\xi^\dagger d\xi - (1-\kappa)d\xi^\dagger\xi\}(1_n + \xi^\dagger\xi)^{-1}]\ ; \quad \kappa \in \boldsymbol{R}\ , \tag{2.16}$$

where the appearance of κ reflects an arbitrariness which does not affect kinematics at all and is usually fixed by taking $\kappa = 1/2$. An action for this Hamiltonian system is thus found as

$$\begin{aligned} S &= \int_{t_1}^{t_2} (\theta_\kappa - F_X dt) \qquad (2.17) \\ &= \int_{t_1}^{t_2} dt\,\mathrm{tr}\left[(1_n + \xi^\dagger\xi)^{-1}\left\{i(\kappa\xi^\dagger\dot\xi - (1-\kappa)\dot\xi^\dagger\xi) - (A + B\xi + \xi^\dagger B^\dagger + \xi^\dagger D\xi)\right\}\right]\ . \end{aligned}$$

Equations of motion then are

$$\begin{aligned} \dot\xi &= i(\xi A - D\xi - B^\dagger + \xi B\xi), \\ \dot\xi^\dagger &= -i(A\xi^\dagger - \xi^\dagger D + \xi^\dagger B^\dagger\xi^\dagger - B). \end{aligned} \tag{2.18}$$

If we put

$$\exp(-iXt) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \gamma(t) & \delta(t) \end{pmatrix}\ , \tag{2.19}$$

we can find the solution:

$$\xi(t) = \{\gamma(t) + \delta(t)\xi(0)\}\{\alpha(t) + \beta(t)\xi(0)\}^{-1}\ . \tag{2.20}$$

The solution (2.20) takes the simplest form in the case of block-diagonal Hamiltonian given by $B = 0$:

$$\xi(t) = U(t)\xi(0)V^\dagger(t), \tag{2.21}$$

where matrices $V(t) \in \mathrm{U}(n)$ and $U(t) \in \mathrm{U}(N-n)$ are given as

$$V(t) = \exp(-iAt),\ U(t) = \exp(-iDt). \tag{2.22}$$

In terms of $\xi(t)$ (2.20) the time dependence of P (2.2) is read as

$$P(t) = \begin{pmatrix} \dfrac{1}{1_n + \xi^\dagger(t)\xi(t)} & \dfrac{1}{1_n + \xi^\dagger(t)\xi(t)}\xi^\dagger(t) \\ \xi(t)\dfrac{1}{1_n + \xi^\dagger(t)\xi(t)} & \xi(t)\dfrac{1}{1_n + \xi^\dagger(t)\xi(t)}\xi^\dagger(t) \end{pmatrix}\ , \tag{2.23}$$

so that

$$P(t) = e^{-iXt}P(0)e^{iXt}\ . \tag{2.24}$$

Therefore we can recognize that the equations of motion (2.18) describe the action of $\mathrm{U}(N)$ on $G_{N,n}$ in a local coordinate system. We can thus see that classical mechanics on $G_{N,n}$ formulated above has a geometric interpretation.

As a final remark note that only $\xi(t) = 0$ $(0 \le t \le T)$ is allowed under the periodic boundary condition $\xi(T) = \xi(0)$ for arbitrary T and X. In the subsequent section the same situation can be found in performing the WKB approximation.

2.2 The D-H formula

In this subsection we discuss the Duistermaat-Heckman localization formula for the classical system defined above. Start with a classical partition function

$$\mathcal{Z}_{cl}(\beta) = \int d\mu(\xi) \exp(-\beta F_H) \ , \quad \beta > 0 \ , \tag{2.25}$$

where

$$d\mu(\xi) \equiv \frac{1}{\{\det(1_n + \xi^\dagger \xi)\}^N} \left(\frac{d\xi d\bar{\xi}}{\pi} \right)^{n(N-n)} \tag{2.26}$$

and satisfies

$$\int d\mu(\xi) = \mathcal{N}(n, N-n) \tag{2.27}$$

with

$$\mathcal{N}(n,p) \equiv \frac{0!1! \cdots (n-1)!}{p!(p+1)! \cdots (p+n-1)!}, \ (p = 0, 1, 2, \ldots) \ , \tag{2.28}$$

(whose verification is postponed until the next section: in (3.48) putting $k \to 0$ we have (2.27).) Here F_H is a Hamiltonian given in terms of a real diagonal matrix,

$$H = \mathrm{diag}(h_1, \ldots, h_n, h_{n+1}, \ldots, h_N), \quad 0 < h_1 < \cdots < h_N, \tag{2.29}$$

to be found as

$$\begin{aligned} F_H &= \mathrm{tr}\{(1_n + \xi^\dagger \xi)^{-1}(H_u + \xi^\dagger H_d \xi)\} \ , \\ H_u &= \mathrm{diag}(h_1, \ldots, h_n), \ H_d = \mathrm{diag}(h_{n+1}, \ldots, h_N) \ . \end{aligned} \tag{2.30}$$

The first task toward the D-H formula is to search critical points of the Hamiltonian by solving

$$\frac{\partial F_H}{\partial \xi_{ia}} = 0, \quad \frac{\partial F_H}{\partial \bar{\xi}_{ia}} = 0, \tag{2.31}$$

which coincide with the right hand sides of the equations of motion (2.18). In view of (2.25) these critical points are nothing but *saddle points* of the integral. For the present case, the saddle point conditions (2.31) become

$$H_u \xi^\dagger - \xi^\dagger H_d = 0, \ \xi H_u - H_d \xi = 0. \tag{2.32}$$

According to the assumption, $h_1 < \cdots < h_N$, to the Hamiltonian (2.30), the conditions cannot be met if $\xi \neq 0$.

Thus we see that $\xi = 0$ is the only one solution of (2.31) *under this parameterization* of $G_{N,n}$, in other words, the unique fixed point in $\mathrm{U}(n) \times \mathrm{U}(N-n)$ (torus, $\mathrm{U}(1)^N$, in this

case) action (2.21). Calculate the second derivative of F_H at $\xi = 0$; the Hessian of the Hamiltonian,

$$\left.\frac{\partial^2 F_H}{\partial \xi_{ia} \partial \bar{\xi}_{jb}}\right|_{\xi=0} = (H_u)_{ab}\delta_{ji} - \delta_{ab}(H_d)_{ji} \ . \tag{2.33}$$

Hence the contribution from this critical point to the saddle point approximation of the integral (2.25) is found to be

$$\begin{aligned} &\int \left(\frac{d\xi d\bar{\xi}}{\pi}\right)^{n(N-n)} \exp\left\{-\beta \operatorname{tr} H_u - \beta \operatorname{tr}(H_d \xi \xi^\dagger - H_u \xi^\dagger \xi)\right\} \\ = \ &\frac{\exp(-\beta \sum_{a=1}^n h_a)}{\beta^{n(N-n)} \prod_{a=1}^n \prod_{i=n+1}^N (h_i - h_a)} \ . \end{aligned} \tag{2.34}$$

As was stressed above we need $\binom{N}{n}$'s local parameterizations to cover the whole phase space, therefore we must consider contributions from other critical points. To this end, recall the unitary transformation given in (2.6). A change of the local parameterization such that

$$P = U(\xi)P_{1,\ldots,n}U^\dagger(\xi) \mapsto P_U = U(\mu_1, \ldots, \mu_n|1, \ldots, n)PU^\dagger(\mu_1, \ldots, \mu_n|1, \ldots, n), \tag{2.35}$$

is equivalent to that of the Hermitian matrix in the Hamiltonian function

$$F_H(P) = \operatorname{tr}(PH) \mapsto F_H(P_U) = F_{H'}(P) = \operatorname{tr}(PH'), \tag{2.36}$$

with

$$H' = U^\dagger(\mu_1, \ldots, \mu_n|1, \ldots, n)HU(\mu_1, \ldots, \mu_n|1, \ldots, n) \ . \tag{2.37}$$

Once recognizing this, we can easily carry out the task; since the new Hamiltonian after the transformation is again diagonal without degeneracy so that a critical point is always located at $\xi = 0$ in each local parameterization. Summing up those contributions, we obtain, as a result of the saddle point approximation,

$$\mathcal{Z}_{cl}(\beta) \simeq \sum_{\mu_1 < \cdots < \mu_n} \frac{\exp(-\beta \sum_{a=1}^n h_{\mu_a})}{\beta^{n(N-n)} \prod_{a=1}^n \prod_{\nu \in \bar{\mu}} (h_\nu - h_{\mu_a})} \ , \tag{2.38}$$

where

$$\bar{\mu} \equiv \{1, \ldots, N\} \backslash \{\mu_1, \ldots, \mu_n\} \ . \tag{2.39}$$

According to the D-H theorem the sum in the right hand side is now *lifted to the exact result.*

To see that this is true, that is, to convince the validity of the D-H theorem, consider, instead of (2.25), the following expression:

$$\int_{\mathrm{H}(n)} d\lambda \int_{M(N,n;\boldsymbol{C})} \left(\frac{dz d\bar{z}}{\pi}\right)^{Nn} \exp\left[-\beta \operatorname{tr}(Z^\dagger H Z) + i \operatorname{tr}\{\lambda(Z^\dagger Z - 1)\}\right] \ , \tag{2.40}$$

where the integration domain of λ is $\mathrm{H}(n)$ and new variables,

$$\begin{aligned}
Z &= \begin{pmatrix} Z_u \\ Z_d \end{pmatrix} , \quad Z \in M(N,n;\boldsymbol{C}) , \\
Z_u &= \begin{pmatrix} z_{1,1} & \cdots & z_{1,n} \\ \vdots & & \vdots \\ z_{n,1} & \cdots & z_{n,n} \end{pmatrix} , \quad Z_d = \begin{pmatrix} z_{n+1,1} & \cdots & z_{n+1,n} \\ \vdots & & \vdots \\ z_{N,1} & \cdots & z_{N,n} \end{pmatrix} ,
\end{aligned} \tag{2.41}$$

have been introduced. An explicit form of λ is given as

$$\lambda = \begin{pmatrix} \lambda_1 & \lambda_{1,2} & \cdots & \lambda_{1,n} \\ \bar{\lambda}_{1,2} & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \lambda_{n-1,n} \\ \bar{\lambda}_{1,n} & \cdots & \bar{\lambda}_{n-1,n} & \lambda_n \end{pmatrix} , \tag{2.42}$$

$$\begin{aligned}
\lambda_a &\in \boldsymbol{R} \quad (a = 1, \ldots, n), \\
\lambda_{a,b} &= \frac{x_{a,b} - i y_{a,b}}{2}, \quad x_{a,b}, y_{a,b} \in \boldsymbol{R}, \quad (1 \leq a < b \leq n),
\end{aligned} \tag{2.43}$$

so that the measure is

$$d\lambda = \prod_{a=1}^{n} \frac{d\lambda_a}{2\pi} \prod_{a<b} \frac{dx_{a,b} dy_{a,b}}{(2\pi)^2}. \tag{2.44}$$

The λ integration gives delta functions which can be solved by means of the following change of variables,

$$Z = \begin{pmatrix} 1_n \\ \xi \end{pmatrix} \frac{1}{\sqrt{1_n + \xi^\dagger \xi}} \zeta; \quad \xi \in M(N-n, n; \boldsymbol{C}), \quad \zeta \in M(n; \boldsymbol{C}) , \tag{2.45}$$

since

$$Z^\dagger Z = \zeta^\dagger \zeta , \tag{2.46}$$

and the measure reads

$$\left(\frac{dz d\bar{z}}{\pi} \right)^{Nn} = \left(\frac{d\xi d\bar{\xi}}{\pi} \right)^{n(N-n)} \frac{1}{\{\det(1_n + \xi^\dagger \xi)\}^N} \left(\frac{d\zeta d\bar{\zeta}}{\pi} \right)^{n^2} \{\det(\zeta^\dagger \zeta)\}^{N-n} . \tag{2.47}$$

(The integration with respect to ζ is easily done convincing that (2.40) is equivalent to (2.25).) Therefore we can recognize that the role of λ is to reduce the number of degrees of freedom from Nn to $Nn - n^2 = n(N-n)$. In this sense λ is called as multiplier and whose coefficient as constraint[23]:

$$\psi_{ab} \equiv (Z^\dagger Z)_{ab} - \delta_{ab} \approx 0 . \tag{2.48}$$

However by exchanging the order of integrations the *Gaussian integrations* with respect to $z_{\mu,a}$'s result in

$$\int\limits_{\mathrm{H}(n)} d\lambda \frac{\exp(-i\,\mathrm{tr}\,\lambda)}{\det(\beta H \otimes 1_n - i 1_N \otimes \lambda^{\mathcal{T}})} \ . \tag{2.49}$$

With the help of the decomposition[24]

$$\lambda = \Omega\lambda_0\Omega^\dagger,\ \ \lambda_0 = \mathrm{diag}(l_1, \ldots, l_n),\ \ \Omega \in \mathrm{SU}(n), \tag{2.50}$$

we find after the Ω integration

$$(2.40) = \int_{-\infty}^{+\infty} \frac{1}{n!} \prod_{a=1}^{n} \frac{dl_a}{2\pi} \prod_{a<b} (l_a - l_b)^2 \frac{\exp(-i\sum_{a=1}^{n} l_a)}{\prod_{a=1}^{n}\prod_{\mu=1}^{N}(\beta h_\mu - i l_a)} \ , \tag{2.51}$$

which leads to, by considering poles and zeros,

$$(2.40) = \sum_{\mu_1 < \cdots < \mu_n} \frac{\exp(-\beta\sum_{a=1}^{n} h_{\mu_a})}{\beta^{n(N-n)}\prod_{a=1}^{n}\prod_{\nu\in\bar{\mu}}(h_\nu - h_{\mu_a})} \ , \tag{2.52}$$

where again $\bar{\mu}$ is given (2.39). This is exactly the same expression as (2.38). Thus *the saddle point approximation (2.38) itself contains the full information of the partition function* $\mathcal{Z}_{cl}$.

We consider in what follows a quantum version of the D-H theorem, that is, the WKB exactness in path integral. Our interpretation to the D-H formula put here will be very helpful in the analysis.

3 Coherent states of $\mathrm{U}(N)$ over $G_{N,n}$

In order to obtain the path integral representation we construct coherent states of $\mathrm{U}(N)$ in this section. We first consider the algebraic method proposed by Perelomov then explore an generalization of the Schwinger boson technique.

3.1 Algebraic construction

The $u(N)$ algebra in terms of generators $E_{\mu\nu}$ reads

$$[E_{\mu\nu}, E_{\rho\sigma}] = \delta_{\nu\rho}E_{\mu\sigma} - \delta_{\mu\sigma}E_{\rho\nu}, \quad (1 \le \mu, \nu, \rho, \sigma \le N). \tag{3.1}$$

First let us build up the coherent state in the fundamental representation. The generators are given

$$\left(E_{\mu\nu}\right)_{\rho\sigma} = \delta_{\mu\rho}\delta_{\nu\sigma} \ . \tag{3.2}$$

Introduce an orthonormal set of basis vectors in $\boldsymbol{C}^N$,

$$(\boldsymbol{e}_\mu)_\nu = \delta_{\mu\nu}, \qquad \boldsymbol{e}_\mu^\dagger \boldsymbol{e}_\nu = \delta_{\mu\nu} \ ; \tag{3.3}$$

to define a fiducial vector, by picking up first n $\boldsymbol{e}_a$ $(a = 1, \cdots, n)$ vectors out of N vectors, such that

$$\vec{\mathcal{E}}_{N,n} = \frac{1}{\sqrt{n!}} \sum_{\sigma \in \mathcal{S}_n} \mathrm{sgn}(\sigma) \boldsymbol{e}_{\sigma(1)} \otimes \cdots \otimes \boldsymbol{e}_{\sigma(n)} \in \boldsymbol{C}^{N^n} \ , \tag{3.4}$$

where $\mathcal{S}_n$ denotes the symmetric group of order n. Consider the map

$$\rho_1 : GL(N;\boldsymbol{C}) \mapsto GL(N^n;\boldsymbol{C}), \ \rho_1(x) \equiv \otimes^n x \left(= \overbrace{x \otimes \cdots \otimes x}^{n} \right) \ , \tag{3.5}$$

then it is obvious

$$d\rho_1(\sum_{a=1}^{n} E_{aa} - \sum_{i=n+1}^{N} E_{ii}) \vec{\mathcal{E}}_{N,n} = n \vec{\mathcal{E}}_{N,n} \ , \tag{3.6}$$

$$d\rho_1(E_{\mu i}) \vec{\mathcal{E}}_{N,n} = 0 \ , \tag{3.7}$$

where

$$\begin{aligned} d\rho_1(E) &\equiv \left. \frac{d}{dt} \right|_{t=0} \rho_1(\exp tE) \\ &= \sum_{p=1}^{n} \otimes^{p-1} 1_N \otimes E \otimes^{n-p} 1_N \ , \end{aligned} \tag{3.8}$$

for $E \in u(N)$. As for this fiducial vector $\vec{\mathcal{E}}_{N,n}$, the following fact should be noted:

Lemma 3.1

$$\rho_1(B) \vec{\mathcal{E}}_{N,n} = \det a \cdot \vec{\mathcal{E}}_{N,n} \ , \tag{3.9}$$

where

$$B \in GL(N;\boldsymbol{C}), \quad B = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \ , \tag{3.10}$$

with

$$a \in GL(n;\boldsymbol{C}), \ b \in M(n, N-n;\boldsymbol{C}), \ c \in GL(N-n;\boldsymbol{C}) \ . \tag{3.11}$$

The proof is obvious so that be omitted. (However some comments would be useful: what this lemma signifies is that $\det a$ ($\vec{\mathcal{E}}_{N,n}$) is an eigenvalue (eigenvector) of $\rho_1(B)$. If we put $n = N$ in (3.9), the relation is nothing but the definition of the determinant.)

Let us now consider an element of SU(N) generated by an orthogonal complement of the Lie algebra of U(n) $\times$ U($N-n$) ,

$$S = \exp\begin{pmatrix} 0 & -\alpha^\dagger \\ \alpha & 0 \end{pmatrix} \quad ; \quad \alpha \in M(N-n, n; \boldsymbol{C}) \ , \tag{3.12}$$

which can be rewritten as

$$S = \begin{pmatrix} \dfrac{1}{\sqrt{1_n + \xi^\dagger\xi}} & -\dfrac{1}{\sqrt{1_n + \xi^\dagger\xi}}\xi^\dagger \\ \xi\dfrac{1}{\sqrt{1_n + \xi^\dagger\xi}} & \dfrac{1}{\sqrt{1_{N-n} + \xi\xi^\dagger}} \end{pmatrix} \quad ; \quad \xi \in M(N-n, n; \boldsymbol{C}) \ , \tag{3.13}$$

with

$$\xi = \alpha\frac{1}{\sqrt{\alpha^\dagger\alpha}}\tan\sqrt{\alpha^\dagger\alpha} \ . \tag{3.14}$$

Noting the Gauss' decomposition $S = LMU$, with

$$L = \begin{pmatrix} 1_n & 0 \\ \xi & 1_{N-n} \end{pmatrix}, \ M = \begin{pmatrix} \dfrac{1}{\sqrt{1_n + \xi^\dagger\xi}} & 0 \\ 0 & \sqrt{1_{N-n} + \xi\xi^\dagger} \end{pmatrix}, \ U = \begin{pmatrix} 1_n & -\xi^\dagger \\ 0 & 1_{N-n} \end{pmatrix}, \tag{3.15}$$

we can obtain a desired (normalized) coherent state:

$$\begin{aligned} |\xi; 1\rangle &\equiv \rho_1(LMU)\vec{\mathcal{E}}_{N,n} \\ &= \frac{1}{\{\det(1_n + \xi^\dagger\xi)\}^{1/2}}\rho_1(L)\,\vec{\mathcal{E}}_{N,n} \ , \end{aligned} \tag{3.16}$$

where we have used the lemma 3.1. While the unnormalized one is given by

$$|\xi; 1) \equiv \rho_1(L)\,\vec{\mathcal{E}}_{N,n}, \qquad (\xi; 1|\xi; 1) = \det(1_n + \xi^\dagger\xi) \ , \tag{3.17}$$

whose second relation can be verified as follows: by noting that $\boldsymbol{L}\boldsymbol{e}_a = \boldsymbol{e}_a + \sum_{i=n+1}^{N}\xi_{ia}\boldsymbol{e}_i$; $(1 \le a \le n)$,

$$|\xi; 1) = \frac{1}{\sqrt{n!}}\sum_{\sigma\,\in\,\mathcal{S}_n}\mathrm{sgn}(\sigma)(\boldsymbol{e}_{\sigma(1)} + \xi_{i_1\sigma(1)}\boldsymbol{e}_{i_1}) \otimes \cdots \otimes (\boldsymbol{e}_{\sigma(n)} + \xi_{i_n\sigma(n)}\boldsymbol{e}_{i_n}) \ . \tag{3.18}$$

Here and for a while a repeated indices implies summation for brevity's sake. Further

$$(\boldsymbol{e}_\sigma^\dagger + \bar{\xi}_{i\sigma}\boldsymbol{e}_i^\dagger)(\boldsymbol{e}_\tau + \eta_{j\tau}\boldsymbol{e}_j) = \delta_{\sigma\tau} + (\xi^\dagger\eta)_{\sigma\tau} \ , \tag{3.19}$$

then

$$(\xi; 1|\eta; 1) = \det(1_n + \xi^\dagger\eta) \ , \tag{3.20}$$

so that

$$\langle\xi; 1|\eta; 1\rangle = \det\left\{(1_n + \xi^\dagger\xi)^{-1/2}(1_n + \xi^\dagger\eta)(1_n + \eta^\dagger\eta)^{-1/2}\right\} \ . \tag{3.21}$$

Next calculate matrix elements of generators: in view of (3.8) the task is to estimate

$$(\xi;1|d\rho_1(E_{\mu\nu})|\eta;1) = \sum_{p=1}^{n}(\xi;1|\otimes^{p-1} 1_N \otimes E_{\mu\nu} \otimes^{n-p} 1_N|\eta;1) \ , \tag{3.22}$$

which can be found, after somewhat lengthy calculations, as

$$\begin{aligned}
(\xi;1|d\rho_1(E_{ab})|\eta;1) &= \det(1_n+\xi^\dagger\eta)\left(\frac{1}{1_n+\xi^\dagger\eta}\right)_{ba} \ , && (3.23)\\
(\xi;1|d\rho_1(E_{ai})|\eta;1) &= \det(1_n+\xi^\dagger\eta)\left(\eta\frac{1}{1_n+\xi^\dagger\eta}\right)_{ia} \ , && (3.24)\\
(\xi;1|d\rho_1(E_{ia})|\eta;1) &= \det(1_n+\xi^\dagger\eta)\left(\frac{1}{1_n+\xi^\dagger\eta}\xi^\dagger\right)_{ai} \ , && (3.25)\\
(\xi;1|d\rho_1(E_{ij})|\eta;1) &= \det(1_n+\xi^\dagger\eta)\left(\eta\frac{1}{1_n+\xi^\dagger\eta}\xi^\dagger\right)_{ji} \ . && (3.26)
\end{aligned}$$

Armed with these, we obtain the matrix element of an arbitrary Hermitian matrix,

$$\begin{aligned}
H &= \sum_{\mu,\nu} h_{\mu\nu}E_{\mu\nu} \\
&= \sum_{a,b} h_{ab}E_{ab} + \sum_{a,i} h_{ai}E_{ai} + \sum_{j,b} h_{jb}E_{jb} + \sum_{i,j} h_{ij}E_{ij} \\
&\equiv \begin{pmatrix} H_{uu} & H_{ud} \\ H_{du} & H_{dd} \end{pmatrix} \ , && (3.27)
\end{aligned}$$

such that

$$(\xi;1|d\rho_1(H)|\eta;1) = (\xi;1|\eta;1)\,\mathrm{tr}\left\{\frac{1}{1_n+\xi^\dagger\eta}(H_{uu}+H_{ud}\eta+\xi^\dagger H_{du}+\xi^\dagger H_{dd}\eta)\right\} \ , \tag{3.28}$$

which is further rewritten to

$$(\xi;1|d\rho_1(H)|\eta;1) = (\xi;1|\eta;1)\,\mathrm{tr}\{P(\xi,\eta)H\} \ , \tag{3.29}$$

or equivalently,

$$\langle\xi;1|d\rho_1(H)|\eta;1\rangle = \langle\xi;1|\eta;1\rangle\,\mathrm{tr}\{P(\xi,\eta)H\} \ , \tag{3.30}$$

where we have introduced a projection $P(\xi,\eta)$,

$$P(\xi,\eta) = \begin{pmatrix} 1_n \\ \eta \end{pmatrix}(1_n+\xi^\dagger\eta)^{-1}\begin{pmatrix} 1_n & \xi^\dagger \end{pmatrix} \ , \tag{3.31}$$

giving

$$\mathrm{tr}\{P(\xi,\eta)H\} = \mathrm{tr}\left\{\frac{1}{1_n+\xi^\dagger\eta}(H_{uu}+H_{ud}\eta+\xi^\dagger H_{du}+\xi^\dagger H_{dd}\eta)\right\} \ . \tag{3.32}$$

(It should be noted that although the definition of $P(\xi,\eta)$ looks singular in the domain $\{(\xi,\eta)|\det(1_n+\xi^\dagger\eta)=0\}$ there is no harm; since the quantity $\mathrm{tr}\{P(\xi,\eta)H\}$ is always accompanied with $\langle\xi;1|\eta;1\rangle$ including $\det(1_n+\xi^\dagger\eta)$ in the numerator.)

Now we generalize the above result to a higher dimensional representation. Consider a tensor product of the coherent state

$$|\xi;1\rangle \mapsto |\xi;k\rangle \equiv \otimes^k|\xi;1\rangle\ ,\ k=0,1,2,\dots\ , \tag{3.33}$$

as well as that of the representation

$$\rho_k(x)\equiv\otimes^k(\rho_1(x)),\ x\in GL(N,\boldsymbol{C})\ , \tag{3.34}$$

to put

$$|\xi;k\rangle=\rho_k(LMU)\vec{\mathcal{E}}^k_{N,n}\ ,\quad \vec{\mathcal{E}}^k_{N,n}\equiv\otimes^k\vec{\mathcal{E}}_{N,n}\ . \tag{3.35}$$

We designate this representation as the k-th representation. The following relations are obvious:

$$d\rho_k(\sum_{a=1}^n E_{aa}-\sum_{i=n+1}^N E_{ii})\vec{\mathcal{E}}^k_{N,n}=kn\vec{\mathcal{E}}^k_{N,n}\ , \tag{3.36}$$

$$d\rho_k(E_{\mu i})\vec{\mathcal{E}}^k_{N,n}=0\ . \tag{3.37}$$

And

(i)

$$\langle\xi;k|\eta;k\rangle=\left[\det\left\{(1_n+\xi^\dagger\xi)^{-1/2}(1_n+\xi^\dagger\eta)(1_n+\eta^\dagger\eta)^{-1/2}\right\}\right]^k\ , \tag{3.38}$$

(ii)

$$\langle\xi;k|d\rho_k(E_{\mu\nu})|\eta;k\rangle=k\,\mathrm{tr}\{P(\xi,\eta)E_{\mu\nu}\}\langle\xi;k|\eta;k\rangle\ , \tag{3.39}$$

hence

$$\langle\xi;k|d\rho_k(H)|\eta;k\rangle=k\,\mathrm{tr}\{P(\xi,\eta)H\}\langle\xi;k|\eta;k\rangle\ . \tag{3.40}$$

In view of these relations, we notice the following facts: (i) the inner product has a form $\{\langle\xi;1|\eta;1\rangle\}^k$ and (ii) the matrix element of a Hamiltonian is proportional to the parameter k, which tells us that the exponent of path integral is proportional to k. Therefore we can perform the $1/k$-expansion when k goes to large like the usual WKB-expansion with respect to $\hbar$.

If we declare that the state $|\xi;k\rangle$ is a coherent state, we must check that the resolution of unity does hold:

$$\mathbf{1}_k=\int d\mu(\xi;k)|\xi;k\rangle\langle\xi;k|\ , \tag{3.41}$$

with

$$d\mu(\xi;k) \equiv \frac{\mathcal{N}(n,k)}{\mathcal{N}(n,N-n+k)} \frac{1}{\{\det(1_n+\xi^\dagger\xi)\}^N} \left(\frac{d\xi d\bar{\xi}}{\pi}\right)^{n(N-n)} , \tag{3.42}$$

where $\mathcal{N}(n,k)$, has been given by (2.28) and $\mathbf{1}_k$ is the identity operator on the representation space. To this end the following formulae are indispensable:

Theorem 3.2 *Let dg be the normalized Haar measure on* $\mathrm{U}(n)$. *Then for $^\forall p \in \boldsymbol{Z}_+$ with $\boldsymbol{Z}_+ = \{0\} \cup \boldsymbol{N}$ and $^\forall X \in M(n;\boldsymbol{C})$, there holds an integration formula*

$$\int_{\mathrm{U}(n)} \frac{dg}{(\det g)^p} \exp\{\mathrm{tr}(gX)\} = \mathcal{N}(n,p)|X|^p, \tag{3.43}$$

$$\mathcal{N}(n,p) \equiv \frac{0!1!\cdots(n-1)!}{p!(p+1)!\cdots(p+n-1)!} . \tag{3.44}$$

Theorem 3.3 *For $^\forall X \in M(n;\boldsymbol{C})$ and $^\forall p \in \boldsymbol{Z}_+$ there holds a differential formula*

$$|\partial_X|\,|X|^p = p(p+1)\cdots(p+n-1)\,|X|^{p-1} , \tag{3.45}$$

where $|X| = \det X$ and $|\partial_X|$ is defined by

$$|\partial_X| \equiv \det\left(\frac{\partial}{\partial x_{ij}}\right) , \quad \text{for } X=(x_{ij}) , \tag{3.46}$$

which is valid even if p is negative integer provided that $X + X^\dagger$ is positive definite and $|X| \neq 0$.

Proofs of these formulae are straightforward but need a bit lengthy calculations therefore we relegate them to Appendix A. (Although the proof of the theorem 3.3, known as Cayley's formula, could be found somewhere, for example, by using the Capelli's identity[25], we supply our own proof for a selfcontained purpose.)

Practically our target is to show instead of (3.41)

$$(\alpha;k|\beta;k) = \int d\mu(\xi;k)\ (\alpha;k|\xi;k\rangle\langle\xi;k|\beta;k) \ ; \quad \alpha,\beta \in M(N-n,n;\boldsymbol{C}) , \tag{3.47}$$

that is,

$$\begin{aligned}
\{\det(1_n+\alpha^\dagger\beta)\}^k &= \int d\mu(\xi;k)\, \frac{\{\det(1_n+\alpha^\dagger\xi)\}^k\{\det(1_n+\xi^\dagger\beta)\}^k}{\{\det(1_n+\xi^\dagger\xi)\}^k} \\
&= \frac{\mathcal{N}(n,k)}{\mathcal{N}(n,N-n+k)} \int \left(\frac{d\xi d\bar{\xi}}{\pi}\right)^{n(N-n)} \frac{1}{\{\det(1_n+\xi^\dagger\xi)\}^N} \\
&\times \frac{\{\det(1_n+\alpha^\dagger\xi)\}^k\{\det(1_n+\xi^\dagger\beta)\}^k}{\{\det(1_n+\xi^\dagger\xi)\}^k} .
\end{aligned} \tag{3.48}$$

Establishing (3.48) is equal to establishing (3.41); since these relations hold for any $|\alpha;k)$ and $|\beta;k)$. In order to accomplish this, there need two others relations.

Corollaly 3.4 *For $^\forall k \in \boldsymbol{Z}_+$, there holds a formula for Gaussian type integration over $M(m+n,n;\boldsymbol{C})$:*

$$\int \left(\frac{dzd\bar{z}}{\pi}\right)^{n(m+n)} |Z^\dagger Z|^k \exp\left\{-\operatorname{tr}(Z^\dagger Z)\right\} = \frac{\mathcal{N}(n,m)}{\mathcal{N}(n,m+k)} \ . \tag{3.49}$$

Proof. By making use of an identity,

$$|Z^\dagger Z|^k \exp\left\{-\operatorname{tr}(Z^\dagger Z)\right\} = (-1)^{nk}|\partial_X|^k\Big|_{X=1_n} \exp\left\{-\operatorname{tr}(XZ^\dagger Z)\right\} \ , \tag{3.50}$$

the left hand side of (3.49) is rewritten as

$$\begin{aligned} & (-1)^{nk}|\partial_X|^k\Big|_{X=1_n} \int \left(\frac{dzd\bar{z}}{\pi}\right)^{n(m+n)} \exp\left\{-\operatorname{tr}(XZ^\dagger Z)\right\} \\ = \ & (-1)^{nk}|\partial_X|^k\Big|_{X=1_n} |X|^{-(m+n)} \ , \end{aligned} \tag{3.51}$$

which becomes, by a repeated use of the formula 3.3,

$$\begin{aligned} (3.51) &= \frac{(m+n)!}{m!} \times \frac{(m+n+1)!}{(m+1)!} \times \cdots \times \frac{(m+n+k-1)!}{(m+k-1)!} \\ &= \frac{\mathcal{N}(n,m)}{\mathcal{N}(n,m+k)} \ . \end{aligned} \tag{3.52}$$

Corollaly 3.5 *For $^\forall p,q \in \boldsymbol{Z}_+$ and $^\forall A,B \in M(m+n,n;\boldsymbol{C})$, there holds*

$$\begin{aligned} & \int_{\mathrm{U}(n)} \frac{dg}{(\det g)^p} \int \left(\frac{dzd\bar{z}}{\pi}\right)^{n(m+n)} |Z^\dagger Z|^q \exp\{-\operatorname{tr}(Z^\dagger Z - Z^\dagger A - gB^\dagger Z)\} \\ = \ & \frac{\mathcal{N}(n,p)\mathcal{N}(n,m+p)}{\mathcal{N}(n,m+p+q)} |B^\dagger A|^p \ . \end{aligned} \tag{3.53}$$

Proof. Integrate with respect to $Z, Z^\dagger$ and follow a similar procedure as above to find

$$(3.53) = (-1)^{nq}|\partial_X|^q\Big|_{X=1_n} \int_{\mathrm{U}(n)} \frac{dg}{(\det g)^p} \exp\{\operatorname{tr}(gB^\dagger AX^{-1})\}|X|^{-(m+n)} \ . \tag{3.54}$$

The formula 3.2 enables us to perform the g integration giving

$$(3.54) = (-1)^{nq}|\partial_X|^q\Big|_{X=1_n} |X|^{-(m+n+p)}\mathcal{N}(n,p)|B^\dagger A|^p \ , \tag{3.55}$$

which turns out, again by the repeated use of the formula 3.3, to be the right hand side of (3.53).

Now we can proceed to our target: first rewrite the left hand side of (3.48) by use of the formula 3.2 as

$$\{\det(1_n+\alpha^\dagger\beta)\}^k = \frac{1}{\mathcal{N}(n,k)}\int_{\mathrm{U}(n)} \frac{dg}{(\det g)^k}\exp\left[\mathrm{tr}\left\{g(1_n+\alpha^\dagger\beta)\right\}\right] , \tag{3.56}$$

whose integrand can be expressed by the Gaussian integration with respect to $Z \in M(N,n;\boldsymbol{C})$,

$$\begin{aligned} &\exp\left[\mathrm{tr}\left\{g(1_n+\alpha^\dagger\beta)\right\}\right] \\ = &\int\left(\frac{dzd\bar{z}}{\pi}\right)^{Nn}\exp\left[-\,\mathrm{tr}\left\{Z^\dagger Z - Z^\dagger\begin{pmatrix}1_n\\ \beta\end{pmatrix} - (1_n,\alpha^\dagger)Zg\right\}\right] . \end{aligned}\tag{3.57}$$

A change of variables, $Z \to (\xi,\zeta)$, as in the previous section from (2.45) to (2.46), gives

$$\begin{aligned} (3.57) \;=\; &\int\left(\frac{d\xi d\bar{\xi}}{\pi}\right)^{n(N-n)}\frac{1}{\{\det(1_n+\xi^\dagger\xi)\}^N}\int\left(\frac{d\zeta d\bar{\zeta}}{\pi}\right)^{n^2}(\det\zeta^\dagger\zeta)^{N-n} \\ &\times\exp\Big[-\,\mathrm{tr}\Big\{\zeta^\dagger\zeta - \zeta^\dagger(1_n+\xi^\dagger\xi)^{-1/2}(1_n+\xi^\dagger\beta) \\ &\qquad -g(1_n+\alpha^\dagger\xi)(1_n+\xi^\dagger\xi)^{-1/2}\zeta\Big\}\Big] . \end{aligned}\tag{3.58}$$

Substituting (3.58) into (3.56) then integrating with respect to ζ and g with the aid of the formula 3.5, we find

$$\begin{aligned} &\text{r.h.s of (3.56)} = \frac{\mathcal{N}(n,k)}{\mathcal{N}(n,N-n+k)} \\ &\times\int\left(\frac{d\xi d\bar{\xi}}{\pi}\right)^{n(N-n)}\frac{1}{\{\det(1_n+\xi^\dagger\xi)\}^N}\frac{\{\det(1_n+\alpha^\dagger\xi)\}^k\{\det(1_n+\xi^\dagger\beta)\}^k}{\{\det(1_n+\xi^\dagger\xi)\}^k} , \end{aligned}\tag{3.59}$$

which is nothing but the right hand side of (3.48).

We now consider another version of coherent state with the aid of the Schwinger boson technique before going into a path integral discussion.

3.2 Coherent state via Schwinger boson

As was stressed in section 2.2, essence of the classical D-H theorem can easily be grasped by increasing degrees of freedom while balancing them with the aid of the multiplier λ: we call such a view point as that of constrained system. If we could find a similar way in a quantum case, then establishment of the localization would be obvious. Fortunately we know such a candidate which might realize our expectation: the method of Schwinger

boson. There in order to obtain a group representation generators of a group are expressed by creation and annihilation operators. The representation space is thus the Fock space whose dimension is too big for a group (especially for a compact one.) Therefore there needs some constraint to reduce the whole space. In a simple case such as $\boldsymbol{C}P^N$[9, 10] it is realized by fixing the total particle number. In this way, the scenario would be hopeful.

Consider operators,

$$a = \begin{pmatrix} a_u \\ a_d \end{pmatrix} \ , \ a_u = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix} \ , \ a_d = \begin{pmatrix} a_{n+1,1} & \cdots & a_{n+1,n} \\ \vdots & & \vdots \\ a_{N,1} & \cdots & a_{N,n} \end{pmatrix} \ , \tag{3.60}$$

which obey $(1 \leq \mu, \nu \leq N; \ 1 \leq a, b \leq n)$

$$[a_{\mu,a}, a^\dagger_{\nu,b}] = \delta_{\mu\nu}\delta_{ab} \ , \ [a_{\mu,a}, a_{\nu,b}] = [a^\dagger_{\mu,a}, a^\dagger_{\nu,b}] = 0 \ . \tag{3.61}$$

In terms of these operators $u(N)$ generators are realized:

$$\hat{E}_{\mu\nu} = \mathrm{tr}(a^\dagger E_{\mu\nu} a) \ ; \tag{3.62}$$

$$[\hat{E}_{\mu\nu}, \hat{E}_{\rho\sigma}] = (\delta_{\nu\rho}\hat{E}_{\mu\sigma} - \delta_{\sigma\mu}\hat{E}_{\rho\nu}) \ . \tag{3.63}$$

The Fock space $\mathcal{F}$ is designated as

$$\mathcal{F} = \mathrm{Span}\{|(n_{\mu,a})\rangle = \prod_{\mu,a} \frac{1}{\sqrt{n_{\mu,a}!}} {a^\dagger_{\mu,a}}^{n_{\mu,a}}|0\rangle \ , \ a_{\mu,a}|0\rangle = 0, \ n_{\mu,a} \in \boldsymbol{Z}_+\} \ . \tag{3.64}$$

Introduce a usual canonical coherent state[3]:

$$|Z\rangle \equiv \exp\left\{\mathrm{tr}(a^\dagger Z - Z^\dagger a)\right\}|0\rangle = \exp\left\{-\frac{1}{2}\mathrm{tr}(Z^\dagger Z)\right\}\exp\left\{\mathrm{tr}(a^\dagger Z)\right\}|0\rangle \ , \tag{3.65}$$

$$\mathbf{1} = \int \left(\frac{dzd\bar{z}}{\pi}\right)^{Nn} |Z\rangle\langle Z|, \tag{3.66}$$

$$\langle Z|Z'\rangle = \exp\left\{-\frac{1}{2}\mathrm{tr}(Z^\dagger Z + Z'^\dagger Z') + \mathrm{tr}(Z^\dagger Z')\right\} \ , \tag{3.67}$$

where $\mathbf{1}$ denotes the identity operator on $\mathcal{F}$ and Z has been given by (2.41).

Let us consider a Hermitian projection operator,

$$P_k \equiv \int \left(\frac{dzd\bar{z}}{\pi}\right)^{Nn} \int_{\mathrm{U}(n)} \frac{dg}{(\det g)^k}|Zg\rangle\langle Z| \ . \tag{3.68}$$

A simple inspection leads to

$$P_k P_{k'} = P_k \delta_{k,k'} \ , \qquad P_k^\dagger = P_k \ . \tag{3.69}$$

In what follows we can recognize that this projection operator indeed reduces $\mathcal{F}$ to the space of the k-th representation in the previous section, but before proceeding it is instructive to discuss how to find the form of P_k as (3.68). By noting

$$\exp\{i\,\mathrm{tr}(\lambda a^\dagger a)\}|Z\rangle = |Zg\rangle, \quad g = e^{i\lambda} \in \mathrm{U}(n)\ , \tag{3.70}$$

so that

$$\frac{1}{(\det g)^k}|Zg\rangle = \exp\left[i\,\mathrm{tr}\left\{\lambda(a^\dagger a - k)\right\}\right]|Z\rangle\ , \tag{3.71}$$

which immediately reminds us of the multiplier part of (2.40) by replacing $a(a^\dagger)$ with $Z(Z^\dagger)$. (However note that the difference of the integration domain of λ; in (2.40) it is infinite but in (3.68) it is bounded. We consider more on this issue in section 5.)

The trace of P_k can be calculated as

$$\begin{aligned}
\mathrm{Tr}\,P_k &= \int\left(\frac{dzd\bar{z}}{\pi}\right)^{Nn}\int_{\mathrm{U}(n)}\frac{dg}{(\det g)^k}\langle Z|Zg\rangle \\
&= \mathcal{N}(n,k)\int\left(\frac{dzd\bar{z}}{\pi}\right)^{Nn}|Z^\dagger Z|^k\exp\left\{-\,\mathrm{tr}(Z^\dagger Z)\right\} \\
&= \frac{\mathcal{N}(n,k)\mathcal{N}(n,N-n)}{\mathcal{N}(n,N-n+k)}\ ,
\end{aligned} \tag{3.72}$$

where we have used (3.67) then the formula 3.4. (We have employed the notation $\mathrm{Tr}(\cdots)$ for the trace over all the Fock space while $\mathrm{Tr}_k(\cdots)$ used below for that within the k-th representation space and those should be distinguished from $\mathrm{tr}(\cdots)$ used for matrix valued quantities.) (3.72) implies the dimension of the k-th representation, that is, the number of independent vectors in $\mathcal{F}$ which satisfy n^2 physical state conditions:

$$(a^\dagger a)_{a,b}|\mathrm{phys}\rangle = k\delta_{a,b}|\mathrm{phys}\rangle\ . \tag{3.73}$$

In partcular, for the case of $k = 1$, the relation (3.72) implies an arbitrariness of choosing a fiducial vector (3.4). It is symmetric with respect to n and $N-n$, which clearly reflects the nature of the base manifold $G_{N,n}$, namely $G_{N,n} \cong G_{N,N-n}$ and is easily checked by an explicit calculation:

$$\frac{\mathcal{N}(n,k)\mathcal{N}(n,N-n)}{\mathcal{N}(n,N-n+k)} = \frac{\mathcal{N}(N-n,k)\mathcal{N}(N-n,n)}{\mathcal{N}(N-n,n+k)}\ . \tag{3.74}$$

Now we show that

$$P_k = \int d\mu(\xi;k)|\xi;k\rangle\langle\xi;k|\ , \tag{3.75}$$

where $|\xi;k\rangle$ is the coherent state of $\mathrm{U}(N)$ over $G_{N,n}$ derived previously but now given by

$$|\xi;k\rangle \equiv \frac{1}{\{\det(1_n+\xi^\dagger\xi)\}^{k/2}}\exp\{\mathrm{tr}(a_d^\dagger\xi a_u)\}\sqrt{\mathcal{N}(n,k)}(\det a_u^\dagger)^k|0\rangle\ . \tag{3.76}$$

Thus we can regard (3.75) as the resolution of unity (3.41).

In order to reach the resolution of unity (3.75) and the coherent state (3.76), first rewrite (3.68) to

$$P_k = \int \left(\frac{dzd\bar{z}}{\pi}\right)^{Nn} \int_{\mathrm{U}(n)} \frac{dg_1 dg_2}{\{\det(g_1 g_2)\}^k} |Zg_1\rangle\langle Zg_2^\dagger| \ , \tag{3.77}$$

which can be recognized directly by putting $Zg_2^\dagger \mapsto Z$ and $g_1 g_2 \mapsto g$ and finally performing the trivial integration with respect to g_2. Then note that from (3.67),

$$\begin{aligned} |Zg\rangle &= \exp\left\{-\frac{1}{2}\,\mathrm{tr}(Z^\dagger Z)\right\} \exp\left\{\mathrm{tr}(a^\dagger Zg)\right\} |0\rangle \\ &= \exp\left\{-\frac{1}{2}\,\mathrm{tr}(Z^\dagger Z)\right\} \exp\left[\mathrm{tr}\{(a_u^\dagger Z_u + a_d^\dagger Z_d)g\}\right] |0\rangle \\ &= \exp\left\{-\frac{1}{2}\,\mathrm{tr}(\zeta^\dagger \zeta)\right\} \exp\left\{\mathrm{tr}(a_d^\dagger \xi a_u)\right\} \exp\left\{\mathrm{tr}(a_u^\dagger \Lambda^{-1/2}\zeta g)\right\} |0\rangle \ , \end{aligned} \tag{3.78}$$

with $\Lambda = 1_n + \xi^\dagger \xi$, where use has been made of the change of variables (2.45) and the Campbell-Baker-Hausdorff formula to the final expression. (As was discussed in the previous section, employing (2.45) is nothing but choosing some fiducial vector.) By means of (3.78) and the formula 3.2 we find

$$\begin{aligned} &\int_{\mathrm{U}(n)} \frac{dg_1}{(\det g_1)^k} |Zg_1\rangle \\ &= \mathcal{N}(n,k) \exp\left\{-\frac{1}{2}\,\mathrm{tr}(\zeta^\dagger \zeta)\right\} \exp\left\{\mathrm{tr}(a_d^\dagger \xi a_u)\right\} \{\det(a_u^\dagger \Lambda^{-1/2}\zeta)\}^k |0\rangle \ . \end{aligned} \tag{3.79}$$

Also note the relation, obtained from the formula 3.4,

$$\int \left(\frac{d\zeta d\bar{\zeta}}{\pi}\right)^{n^2} |\zeta^\dagger \zeta|^{N-n+k} \exp\left\{-\,\mathrm{tr}(\zeta^\dagger \zeta)\right\} = \frac{1}{\mathcal{N}(n, N-n+k)} \ . \tag{3.80}$$

Substituting (3.79) (and whose conjugate) into (3.77) then utilizing (3.80) , we finally arrive at

$$\begin{aligned} P_k &= \frac{\mathcal{N}(n,k)^2}{\mathcal{N}(n,N-n+k)} \int \left(\frac{d\xi d\bar{\xi}}{\pi}\right)^{n(N-n)} \frac{1}{\{\det(1_n + \xi^\dagger \xi)\}^{N+k}} \\ &\quad \times \exp\left\{\mathrm{tr}(a_d^\dagger \xi a_u)\right\} (\det a_u^\dagger)^k |0\rangle\langle 0| (\det a_u)^k \exp\left\{\mathrm{tr}(a_u^\dagger \xi^\dagger a_d)\right\} \ . \end{aligned} \tag{3.81}$$

Therefore we have established (3.75) and (3.76). By comparing (3.76) with (3.16) and (3.33) the state $\sqrt{\mathcal{N}(n,k)}(\det a_u^\dagger)^k|0\rangle$ should be identified to $\vec{\mathcal{E}}^k_{N,n}$. Therefore the projection operator onto the subspace can now be regarded as the resolution of unity in the space of the k-th representation.

4 Path integral and WKB

In this section we first build up a path integral representation of a character formula by means of the coherent states developed previously. We then perform the WKB approximation to find the result is exact. The mechanism of the exactness is uncovered by reformulating the theory in terms of the generalized Schwinger boson technique.

4.1 Path integral representation

Take a Hamiltonian in the k-th representation

$$\hat{H} = d\rho_k(H)\ ,\ \ H = \mathrm{diag}(h_1, \ldots, h_N),\ \ h_\mu \in \boldsymbol{R},\ \ h_1 < \cdots < h_N\ . \tag{4.1}$$

Consider the trace of the time evolution operator which is from now on designated as the *character formula* of the k-th representation:

$$\begin{aligned} \mathcal{Z}_k(T) &\equiv \mathrm{Tr}_k\, \rho_k(e^{-iHT}) \\ &= \lim_{M\to\infty} \int d\mu(\xi;k)\langle \xi;k|\left\{d\rho_k(1_N - i\Delta t H)\right\}^M |\xi;k\rangle\ , \end{aligned} \tag{4.2}$$

where $\Delta t = T/M$. Inserting the resolution of unity (3.41) into the final expression successively, we obtain

$$\mathcal{Z}_k(T) = \lim_{M\to\infty} \int_{\mathrm{PBC}} \prod_{j=1}^{M} d\mu(\xi(j);k)\langle \xi(j);k|d\rho_k(1_N - i\Delta t H)|\xi(j-1);k\rangle\ , \tag{4.3}$$

where as before "PBC" means $\xi(0) = \xi(M)$. In view of (3.40),

$$\begin{aligned} &\langle \xi(j);k|d\rho_k(1_N - i\Delta t H)|\xi(j-1);k\rangle \\ =\ &\langle \xi(j);k|\xi(j-1);k\rangle\, [1 - ik\Delta t\, \mathrm{tr}\{P(\xi(j),\xi(j-1))H\}] \\ =\ &\langle \xi(j);k|\xi(j-1);k\rangle \exp\left[-ik\Delta t\, \mathrm{tr}\{P(\xi(j),\xi(j-1))H\}\right] \\ &\times \left\{1 + O((\Delta t)^2)\right\}\ . \end{aligned} \tag{4.4}$$

Employing the expression (3.38) for $\langle \xi(j);k|\xi(j-1);k\rangle$, we obtain

$$\begin{aligned} \mathcal{Z}_k(T) = &\lim_{M\to\infty} \int_{\mathrm{PBC}} \prod_{j=1}^{M} d\mu(\xi(j);k) \\ &\times \exp\left[-k\sum_{i=1}^{M} \mathrm{tr}\left\{\log(1_n + \xi^\dagger(i)\xi(i)) - \log(1_n + \xi^\dagger(i)\xi(i-1))\right\}\right] \\ &\times \exp\left[-ik\Delta t \sum_{j=1}^{M} \mathrm{tr}\{P(\xi(j),\xi(j-1))H\}\right]\ , \end{aligned} \tag{4.5}$$

where we have discarded terms of $O((\Delta t)^2)$, whose fact also brings us to

$$\begin{aligned}\mathcal{Z}_k(T) &= (\det V(T))^k \lim_{M\to\infty} \int_{\text{PBC}} \prod_{j=1}^{M} d\mu(\xi(j);k) \\ &\quad \times \exp\left[-k\sum_{i=1}^{M} \operatorname{tr}\left\{ \log(1_n + \xi^\dagger(i)\xi(i)) \right.\right. \\ &\qquad\qquad \left.\left. - \log(1_n + \xi^\dagger(i)U(\Delta t)\xi(i-1)V^\dagger(\Delta t))\right\}\right] , \qquad (4.6)\end{aligned}$$

where $U(t)$ and $V(t)$ have been defined by (2.22),

$$U(t) = e^{-iH_d t} \in \mathrm{U}(N-n) \ , \ V(t) = e^{-iH_u t} \in \mathrm{U}(n) \ , \qquad (4.7)$$

$$H_u = \operatorname{diag}(h_1, \ldots, h_n) \ , \ H_d = \operatorname{diag}(h_{n+1}, \ldots, h_N) \ . \qquad (4.8)$$

By noting that

$$\begin{aligned}& \exp\left[-k \operatorname{tr}\left\{\log(1_n + \xi^\dagger(i)\xi(i)) - \log(1_n + \xi^\dagger(i)U(\Delta t)\xi(i-1)V^\dagger(\Delta t))\right\}\right] \\ = & \left\{\frac{\det(1_n + \xi^\dagger(i)U(\Delta t)\xi(i-1)V^\dagger(\Delta t))}{\det(1_n + \xi^\dagger(i)\xi(i))}\right\}^k , \qquad (4.9)\end{aligned}$$

(4.6) becomes

$$\begin{aligned}\mathcal{Z}_k(T) &= (\det V(T))^k \lim_{M\to\infty} \int_{\text{PBC}} \prod_{i=1}^{M} d\mu(\xi(i);k) \\ &\qquad \times \prod_{j=1}^{M} \langle \xi(j)V(\Delta t);k | U(\Delta t)\xi(j-1);k\rangle , \qquad (4.10)\end{aligned}$$

where use has been made of (3.38). The multiple integration can be carried out with the aid of (3.48) to yield

$$\mathcal{Z}_k(T) = (\det V(T))^k \int d\mu(\xi;k) \left\{\frac{\det(1_n + \xi^\dagger U(T)\xi V^\dagger(T))}{\det(1_n + \xi^\dagger\xi)}\right\}^k . \qquad (4.11)$$

(See a further discussion in Appendix B.) By means of of the formula 3.2 the final integration can be performed, however the task is postponed for a while.

4.2 The WKB approximation

Let us examine the WKB approximation of the path integral expression (4.6). Equations of motion are

$$\begin{aligned}& \xi(j)\{1_n + \xi^\dagger(j)\xi(j)\}^{-1} \\ = \ & U(\Delta t)\xi(j-1)V^\dagger(\Delta t)\{1_n + \xi^\dagger(j)U(\Delta t)\xi(j-1)V^\dagger(\Delta t)\}^{-1} , \qquad (4.12)\end{aligned}$$

$$
\begin{aligned}
& \{1_n + \xi^\dagger(j)\xi(j)\}^{-1}\xi^\dagger(j) \\
= \; & \{1_n + V^\dagger(\Delta t)\xi^\dagger(j+1)U(\Delta t)\xi(j)\}^{-1}V^\dagger(\Delta t)\xi^\dagger(j+1)U(\Delta t) \ . \qquad (4.13)
\end{aligned}
$$

Under the present situation, that is, calculating the character, solutions should meet the periodic boundary condition $\xi(0) = \xi(M)$. Clearly, only $\xi(j) = 0$ for all $1 \le j \le M$ can fulfill the condition. Therefore, by putting $\xi = z/\sqrt{k}$ and noting

$$
\frac{\mathcal{N}(n,k)}{\mathcal{N}(n,N-n+k)} \overset{k\to\infty}{\sim} k^{n(N-n)} \ , \tag{4.14}
$$

the dominant contribution from this classical solution is read as

$$
\begin{aligned}
\tilde{\mathcal{Z}}_k(T) \overset{k\to\infty}{\sim} & \lim_{M\to\infty} \int_{\text{PBC}} \prod_{j=1}^{M} \left(\frac{dz(j)d\bar{z}(j)}{\pi} \right)^{n(N-n)} \\
& \times \exp\left[-\sum_{j=1}^{M} \mathrm{tr}\, z^\dagger(j)\{z(j) - U(\Delta t)z(j-1)V^\dagger(\Delta t)\} \right] \ , \qquad (4.15)
\end{aligned}
$$

where

$$
\tilde{\mathcal{Z}}_k(T) \equiv \frac{\mathcal{Z}_k(T)}{(\det V(T))^k} \ . \tag{4.16}
$$

To perform the integration, it is convenient to utilize the Fourier transformation which respects "PBC":

$$
z(j) = \sum_{r=0}^{M-1} \frac{1}{\sqrt{M}} e^{-2\pi i j r/M} \tilde{z}(r) \ , \quad \tilde{z}(r) \in M(N-n, n; \boldsymbol{C}), \tag{4.17}
$$

and enables us to write

$$
\begin{aligned}
& \sum_{j=1}^{M} z^\dagger(j)\{z(j) - U(\Delta t)z(j-1)V^\dagger(\Delta t)\} \\
= \; & \sum_{r=0}^{M-1} \tilde{z}^\dagger(r)\{\tilde{z}(r) - U(\Delta t)\tilde{z}(r)V^\dagger(\Delta t)e^{2\pi i r/M}\} \ . \qquad (4.18)
\end{aligned}
$$

Since the Jacobian is trivial the integration with respect to $\tilde{z}$ can readily be performed to give

$$
\tilde{\mathcal{Z}}_k(T) = \lim_{M\to\infty} \frac{1}{\prod_{r=0}^{M-1} \det\{1_{N-n} \otimes 1_n - e^{2\pi i r/M} U(\Delta t) \otimes \bar{V}(\Delta t)\}}, \tag{4.19}
$$

where $\bar{V}$ is the complex conjugate of V. Recalling the identity which holds for any $X \in M(m; \boldsymbol{C})$,

$$
\prod_{r=0}^{M-1} (1_m - e^{2\pi i r/M} X) = 1_m - X^M \ , \tag{4.20}
$$

we finally obtain

$$\begin{aligned}\tilde{\mathcal{Z}}_k(T) &= \frac{1}{\det\{1_{N-n}\otimes 1_n - U(T)\otimes \bar{V}(T)\}} \\ &= \frac{1}{\prod_{i=n+1}^{N}\prod_{a=1}^{n}\{1-e^{-i(h_i-h_a)T}\}}\ . \end{aligned} \tag{4.21}$$

Similar to the classical case there are $\binom{N}{n}$'s classical solutions in total. Therefore taking all the contributions and going back to the relation (4.16), we obtain

$$\mathcal{Z}_k(T) \overset{k\to\infty}{\sim} \sum_{\mu_1<\cdots<\mu_n} \frac{\exp\left(-ik\sum_{a=1}^{n} h_{\mu_a}T\right)}{\prod_{a=1}^{n}\prod_{\nu\in\bar{\mu}}\{1-e^{-i(h_\nu-h_{\mu_a})T}\}}\ , \tag{4.22}$$

with $\bar{\mu}$ being given by (2.39).

If we notice that the right hand side of (4.22) is rewritten, by means of the Laplace expansion of determinant, as

$$\text{r.h.s of (4.22)} = \frac{|\epsilon^{N-1+k},\ldots,\epsilon^{N-n+k},\epsilon^{N-n-1},\ldots,\epsilon^{1},1|}{|\epsilon^{N-1},\ldots,\epsilon^{1},1|}\ , \tag{4.23}$$

where

$$|\epsilon^{m_1},\ldots,\epsilon^{m_N}| \equiv \begin{vmatrix} \epsilon_1^{m_1} & \cdots & \epsilon_1^{m_N} \\ \vdots & \vdots & \vdots \\ \epsilon_N^{m_1} & \cdots & \epsilon_N^{m_N} \end{vmatrix}\ , \quad \epsilon_\mu \equiv e^{-ih_\mu T}, \tag{4.24}$$

we can conclude that the WKB approximation gives an exact result in the case of the path integral expression (4.5); since (4.23) is nothing but the Weyl character formula[26] of $\mathrm{U}(N)$ for the present representation. (It will be evident how to obtain a similar expression for the classical counter part (2.38).) The mechanism of this exactness can be uncovered by means of the Schwinger boson technique.

4.3 The mechanism of exactness

In view of (4.6), or even after $M-1$ integrations as in (4.11), the remaining integration looks still hard to perform. We know, however, another recipe for constructing a path integral expression with the aid of the generalized Schwinger boson.

As was introduced in (3.62), any element $E_{\mu\nu}\in u(N)$ has an operator counterpart on $\mathcal{F}$. Since any $N\times N$ Hermitian matrix is expandable in terms of these basis matrices, a quantum Hamiltonian is given as a self-adjoint operator on $\mathcal{F}$:

$$\hat{X} \equiv \mathrm{tr}(a^\dagger X a)\ ,\ X\in \mathrm{H}(N). \tag{4.25}$$

Taking the diagonal matrix H in (4.1) we define $\hat{H}$ to interpret the character formula (4.2) as

$$\mathscr{Z}_k(T) = \mathrm{Tr}_k \exp(-i\hat{H}T) \ , \tag{4.26}$$

in this new formulation. If we use the resolution of unity (3.75), we arrive exactly at the same expression as (4.2). However, recall that the operator P_k is not only the resolution of unity in the k-th representation space but also the projection operator onto the subspace of the Fock space. Therefore rewrite (4.26) as

$$\begin{aligned} \mathscr{Z}_k(T) &= \mathrm{Tr}\{\exp(-i\hat{H}T)P_k\} \\ &= \int_{\mathrm{U}(n)} \frac{dg}{(\det g)^k} \int \left(\frac{dzd\bar{z}}{\pi}\right)^{Nn} \langle Z|e^{-i\hat{H}T}|Zg_\varepsilon\rangle \ , \end{aligned} \tag{4.27}$$

where we have introduced a regularization parameter ε, being put zero after all:

$$g \mapsto g_\varepsilon = e^{-\varepsilon} g \tag{4.28}$$

which can legitimatize the exchange of order of integrations. Apart from the g-integration, the path integral expression for the right hand side of (4.27) can be found straightforwardly: divide the time duration into M segments and insert the resolution of unity (3.66) successively to obtain

$$\begin{aligned} &\int \left(\frac{dzd\bar{z}}{\pi}\right)^{Nn} \langle Z|e^{-i\hat{H}T}|Zg_\varepsilon\rangle \\ &= \lim_{M\to\infty} \int_{\mathrm{TBC}} \prod_{i=1}^{M} \left(\frac{dz(i)d\bar{z}(i)}{\pi}\right)^{Nn} \prod_{j=1}^{M} \langle Z(j)|(\mathbf{1} - i\hat{H}\Delta t)|Z(j-1)\rangle \ , \end{aligned} \tag{4.29}$$

with "TBC" being a twisted boundary condition $Z(0) = Z(M)g_\varepsilon$. Since the Hamiltonian is bilinear with respect to $a^\dagger$ and a, it is a simple task to reach

$$\begin{aligned} (4.29) \ &= \lim_{M\to\infty} \int_{\mathrm{TBC}} \prod_{i=1}^{M} \left(\frac{dz(i)d\bar{z}(i)}{\pi}\right)^{Nn} \\ &\quad \times \exp\left[-\sum_{j=1}^{M} \mathrm{tr}\, Z^\dagger(j)\left\{Z(j) - (1_N - iH\Delta t)Z(j-1)\right\}\right] \ . \end{aligned} \tag{4.30}$$

In order to carry out the Gaussian path integral in this case, follow a similar procedure from (4.17) to (4.21) except employing the Fourier transformation met with "TBC":

$$z(j) = \sum_{r=0}^{M-1} \frac{1}{\sqrt{M}} e^{-2\pi i jr/M} \tilde{z}(r)\,(g_\varepsilon)^{j/M} \ . \tag{4.31}$$

Therefore

$$(4.27) = \int_{\mathrm{U}(n)} \frac{dg}{(\det g)^k} \frac{1}{\det(1_N \otimes 1_n - e^{-iHT} \otimes g_\varepsilon^{\mathcal{T}})} \ . \tag{4.32}$$

The remaining g-integration can be performed by use of the decomposition[27, 24]

$$g = \Omega g_0 \Omega^\dagger,\ \ g_0 = \mathrm{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n}),\ \ \Omega \in \mathrm{SU}(n), \tag{4.33}$$

then by the integration with respect to Ω giving

$$(4.32) = \int_0^{2\pi} \left(\frac{d\theta}{2\pi}\right)^n \frac{1}{n!} \prod_{a<b} |e^{i\theta_a} - e^{i\theta_b}|^2 \frac{\exp\left(-ik\sum_{a=1}^n \theta_a\right)}{\prod_{\mu=1}^N \prod_{a=1}^n (1 - e^{-ih_\mu T + i\theta_a - \varepsilon})}\ . \tag{4.34}$$

By putting $w_a = e^{-i\theta_a}$, the integration over the maximal torus is converted into a multiple contour integrations:

$$(4.34) = \frac{1}{n!} \oint \left(\frac{dw}{2\pi i}\right)^n \prod_{a\neq b} (w_a - w_b) \frac{\prod_{a=1}^n w_a^{N-n+k}}{\prod_{\mu=1}^N \prod_{a=1}^n (w_a - e^{-ih_\mu T - \varepsilon})}\ . \tag{4.35}$$

Taking into account all the contributions from poles, we finally obtain

$$\mathcal{Z}_k(T) = \sum_{\mu_1<\cdots<\mu_n} \frac{\exp\left(-ik\sum_{a=1}^n h_{\mu_a} T\right)}{\prod_{a=1}^n \prod_{\nu\in\bar{\mu}} \{1 - e^{-i(h_\nu - h_{\mu_a})T}\}}\ , \tag{4.36}$$

where again we have used the notation (2.39). (4.36) exactly matches with (4.22). Hence the WKB approximation is exact in path integral for the character formula (4.2) which is now interpreted as that of $\mathrm{U}(N)$ represented over $G_{N,n}$.

Here, the reason is quite obvious; since the path integral representation, in view of (4.30), is essentially Gaussian with an additional g-integration which is regarded as imposing the physical state condition. Evidently there is no room for the appearance of k^{-1}. As is stated in [14], we may conclude that the path integral expression we have discussed is kinematically nonlinear but dynamically free. (The situation would correspond to a free field over nontrivial phase space, which should compare to the harmonic oscillator (free field!) over a flat phase space.)

5 Discussion

In this paper we have clarified the exactness of the WKB approximation for the $\mathrm{U}(N)$ character formula which is formulated by path integral over $G_{N,n}$. We have employed a time slicing method and coherent states to build up a path integral representation. We have made two different approaches: Perelomov's generalized coherent state and a generalized method of Schwinger boson. In terms of the latter method, that is, a view from a constrained system clarifies the reason for exactness: both cases, classical (2.38) as well as (4.22), can be interpreted such that the targets, the classical partition function (2.25) and the character formula (4.2), have essentially been expressed as Gaussian forms.

Note, however, the difference: while critical points are controlled by eigenvalues of the Hermitian matrix, (2.32) or (2.38) in the classical case, but those are controlled by those of the unitary matrix, (4.12), (4.13) or (4.22) in the quantum case. This originates from the difference in the form of constrains: in the classical case we can naively put constraints (2.48) into a trivial partition function in terms of the delta function to obtain (2.40), in other words, the integration domain of the multiplier λ is infinite but in the quantum case, as can be seen from (3.68), it is *compact*.

The compactness of the integration domain is indispensable in the quantum case: to see the situation more clearly let us examine the following model. Take the $\boldsymbol{C}P^1$ case as a simple example:

$$H = \boldsymbol{z}^\dagger h \boldsymbol{z} \ , \quad h = \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix} \in \mathrm{H}(2) \ , \quad \boldsymbol{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \boldsymbol{C}^2 \ , \tag{5.1}$$

with a constraint

$$\psi \equiv \boldsymbol{z}^\dagger \boldsymbol{z} - p \approx 0, \quad p \in \boldsymbol{R}_+. \tag{5.2}$$

The fundamental Poisson brackets are given by

$$\{z_\mu, \bar{z}_\nu\} = -i\delta_{\mu\nu}, \ \{z_\mu, z_\nu\} = \{\bar{z}_\mu, \bar{z}_\nu\} = 0, \quad (\mu, \nu = 1, 2) \ . \tag{5.3}$$

In order to reach the reduced manifold $\boldsymbol{C}P^1$ we need an additional constraint to fix one phase of complex numbers, say z_1. To this end, a change of variables

$$\boldsymbol{z} = \begin{pmatrix} 1 \\ \xi \end{pmatrix} \frac{1}{\sqrt{1+|\xi|^2}} \zeta \ , \tag{5.4}$$

is utilized. The constraint (5.2) is read as $\psi = |\zeta|^2 - p$ and the desired one is found as

$$\chi = \frac{1}{2i} \log(\zeta/\bar{\zeta}) - \phi_0, \quad 0 \le \phi_0 < 2\pi \ . \tag{5.5}$$

They satisfy

$$\{\psi, \chi\} = 1 \ , \tag{5.6}$$

so that the Dirac brackets can be constructed giving

$$\{\xi, \bar{\xi}\}_{\mathrm{D}} = -\frac{i}{p}(1+|\xi|^2)^2. \tag{5.7}$$

In this way classical mechanics on the reduced phase space can be obtained without any problems.

A prescription to quantum theory would be found by means of path integral developed by Faddeev-Senjanovic(FS)[28]:

$$\begin{aligned} \mathcal{Z}_p^{(\mathrm{FS})} \ &\equiv \ \lim_{M\to\infty} \int_{\mathrm{PBC}} \prod_{i=1}^{M} \frac{(dz(i)d\bar{z}(i))^2}{\pi} \delta(\psi(i))\delta(\chi(i)) \\ &\times \exp\left[i \sum_{j=1}^{M} \left\{ i\boldsymbol{z}^\dagger(j)\Delta\boldsymbol{z}(j) - \Delta t \boldsymbol{z}^\dagger(j) h \boldsymbol{z}(j-1) \right\} \right] \ , \end{aligned} \tag{5.8}$$

where

$$\Delta \boldsymbol{z}(j) \equiv \boldsymbol{z}(j) - \boldsymbol{z}(j-1) \ . \tag{5.9}$$

Although the way of finding (5.8) is rather heuristic, the result seems convincing provided constrained systems are given in the configuration space such as the sphere[29]. Therefore we employ this as a starting point of quantum theory.

With the aid of the change of variables (5.4), (5.8) becomes

$$\begin{aligned}
\mathcal{Z}_p^{(\mathrm{FS})} &= \lim_{M\to\infty} \int_{\mathrm{PBC}} \prod_{i=1}^{M} \frac{d\xi(i)d\bar{\xi}(i)}{\pi(1+|\xi(i)|^2)^2} d\zeta(i)d\bar{\zeta}(i)|\zeta(i)|^2\delta(\psi(i))\delta(\chi(i)) \\
&\quad \times \exp\left[i\sum_{j=1}^{M} \left\{ i|\zeta(j)|^2 - i\bar{\zeta}(j)\zeta(j-1)\frac{1+\bar{\xi}(j)\xi(j-1)}{(1+\bar{\xi}(j)\xi(j))^{1/2}(1+\bar{\xi}(j-1)\xi(j-1))^{1/2}} \right.\right. \\
&\quad \left.\left. -\Delta t\bar{\zeta}(j)\zeta(j-1)\frac{a+b\xi(j-1)+\bar{\xi}(j)\bar{b}+\bar{\xi}(j)d\xi(j-1)}{(1+\bar{\xi}(j)\xi(j))^{1/2}(1+\bar{\xi}(j-1)\xi(j-1))^{1/2}} \right\} \right] \ .
\end{aligned} \tag{5.10}$$

A trivial integration with respect to ζ leads to

$$\begin{aligned}
\mathcal{Z}_p^{(\mathrm{FS})} &= \lim_{M\to\infty} \int_{\mathrm{PBC}} \prod_{i=1}^{M} \frac{p d\xi(i)d\bar{\xi}(i)}{\pi(1+|\xi(i)|^2)^2} \\
&\quad \times \exp\left[-p\sum_{j=1}^{M} \left\{ 1 - \frac{1+\bar{\xi}(j)\xi(j-1)}{(1+\bar{\xi}(j)\xi(j))^{1/2}(1+\bar{\xi}(j-1)\xi(j-1))^{1/2}} \right.\right. \\
&\quad \left.\left. +i\Delta t\frac{a+b\xi(j-1)+\bar{\xi}(j)\bar{b}+\bar{\xi}(j)d\xi(j-1)}{(1+\bar{\xi}(j)\xi(j))^{1/2}(1+\bar{\xi}(j-1)\xi(j-1))^{1/2}} \right\} \right] \ ,
\end{aligned} \tag{5.11}$$

which should be compared with the correct one, $\boldsymbol{C}P^1$ version of (4.5),

$$\begin{aligned}
\mathcal{Z}_k(T) &= \lim_{M\to\infty} \int_{\mathrm{PBC}} \prod_{i=1}^{M} \frac{(k+1)d\xi(i)d\bar{\xi}(i)}{\pi(1+|\xi(i)|^2)^2} \\
&\quad \times \exp\left[-k\sum_{j=1}^{M} \left\{ \log(1+|\xi(j)|^2) - \log(1+\bar{\xi}(j)\xi(j-1)) \right.\right. \\
&\quad \left.\left. + i\Delta t\frac{a+b\xi(j-1)+\bar{\xi}(j)\bar{b}+\bar{\xi}(j)d\xi(j-1)}{1+\bar{\xi}(j)\xi(j-1)} \right\} \right] \ .
\end{aligned} \tag{5.12}$$

In view of these, even if an arbitrary parameter p in (5.11) would be set to $k \in \boldsymbol{Z}_+$, a failure of the FS prescription for the present model is now obvious. Nevertheless a formal continuum limit of (5.11) seems reasonably geometric and respect the classical feature of the system. First rely on a naive expansion,

$$\xi(j-1) \sim \xi(j) - \Delta t\dot{\xi}(j) \ , \tag{5.13}$$

which brings (5.11) to

$$(5.11) \quad \to \quad \int_{\text{PBC}} \prod_{0\le t\le T} \frac{pd\xi(t)d\bar{\xi}(t)}{\pi(1+|\xi(t)|^2)^2}$$

$$\times \exp\left[ip\int_0^T dt\left\{\frac{i}{2}\frac{\bar{\xi}\dot{\xi}-\dot{\bar{\xi}}\xi}{1+|\xi|^2} - \frac{1}{1+|\xi|^2}(a+b\xi+\bar{b}\bar{\xi}+\bar{\xi}d\xi)\right\}\right] , \qquad (5.14)$$

whose exponent consists of (classical) $\boldsymbol{C}P^1$ action. (Of course, we can arrive also at (5.14), starting from (5.12) with an replacement of $k+1$ in the measure by p and taking the same limit.)

We should, therefore, discard or modify the expression (5.8) in order to find a correct quantum theory. As was stated above, to see the importance of the compactness of the multiplier we employ a modified expression:

$$\mathcal{Z}_p^{(\text{FS-I})} \equiv \lim_{M\to\infty}\int_{\text{PBC}}\prod_{i=1}^{M}\left(\frac{dz(i)d\bar{z}(i)}{\pi}\right)^2\int_{-\infty}^{\infty}\frac{\Delta t}{2\pi}d\lambda(i)\exp\left[i\sum_{j=1}^{M}\left\{i\boldsymbol{z}^\dagger(j)\Delta\boldsymbol{z}(j)\right.\right.$$

$$\left.\left.-\Delta t\boldsymbol{z}^\dagger(j)h\boldsymbol{z}(j-1)+\Delta t\lambda(j)(\boldsymbol{z}^\dagger(j)\boldsymbol{z}(j-1)-p)\right\}\right] . \qquad (5.15)$$

Here the χ constraints have simply been discarded while the ψ constraints now read as $\boldsymbol{z}^\dagger(j)\boldsymbol{z}(j-1)-p\approx 0$ and have been Fourier-transformed in (5.8). Note that λ plays a role of the multiplier and still travels an *infinite range*. If we notice that a change of variables

$$\boldsymbol{z}(j) = \boldsymbol{z}'(j)\exp\left\{i\Delta t\sum_{k=1}^{j}\lambda(k)\right\} , \qquad (5.16)$$

$$\lambda(j) = \sum_{r=0}^{M-1}\frac{1}{\sqrt{M}}e^{-2\pi ijr/M}\tilde{\lambda}(r) , \qquad (5.17)$$

wipes out almost all $\tilde{\lambda}(r)$ leaving only $\tilde{\lambda}(0)$(constant mode of $\lambda(j)$) in the integrand we further modify (5.8), by throwing away infinities from $\tilde{\lambda}(r)$'s, to

$$\mathcal{Z}_p^{(\text{FS-II})} \equiv \lim_{M\to\infty}\int_{\text{TBC}}\prod_{i=1}^{M}\left(\frac{dz(i)d\bar{z}(i)}{\pi}\right)^2\int_{-\infty}^{\infty}\frac{d\lambda}{2\pi}e^{-ip\lambda}$$

$$\times\exp\left[i\sum_{j=1}^{M}\left\{i\boldsymbol{z}^\dagger(j)\Delta\boldsymbol{z}(j)-\Delta t\boldsymbol{z}^\dagger(j)h\boldsymbol{z}(j-1)\right\}\right] , \qquad (5.18)$$

where as before "TBC" denotes $\boldsymbol{z}(0)=\boldsymbol{z}(M)e^{i\lambda}$ and all primes have been removed. Now the Gaussian integrations with respect to $\boldsymbol{z}$'s can be done, by introducing a regularization parameter $\varepsilon>0$, to yield

$$\mathcal{Z}_p^{(\text{FS-II})} = \lim_{\varepsilon\to\infty}\int_{-\infty}^{+\infty}\frac{d\lambda}{2\pi}e^{-ip\lambda}\frac{1}{(1-e^{-ih_1T+i\lambda-\varepsilon})(1-e^{-ih_2T+i\lambda-\varepsilon})} , \qquad (5.19)$$

where h_i's are eigenvalues of h in (5.1). In view of (5.19), p must be some positive integer, otherwise the result is zero, which leads us furthermore to the conclusion that the integration domain of λ in (5.19) *must be replaced by a compact one* $0 \leq \lambda \leq 2\pi$. (Since otherwise we obtain infinite copies of the same integration.) Therefore we should have

$$\mathcal{Z}_p^{(\text{correct})} = \lim_{\varepsilon \to \infty} \int_0^{2\pi} \frac{d\lambda}{2\pi} e^{-ip\lambda} \frac{1}{(1 - e^{-ih_1 T + i\lambda - \varepsilon})(1 - e^{-ih_2 T + i\lambda - \varepsilon})} , \tag{5.20}$$

that is

$$\begin{aligned} \mathcal{Z}_p^{(\text{correct})} &= \lim_{M \to \infty} \int_{\text{TBC}} \prod_{i=1}^{M} \left(\frac{dz(i) d\bar{z}(i)}{\pi} \right)^2 \int_0^{2\pi} \frac{d\lambda}{2\pi} e^{-ip\lambda} \\ &\quad \times \exp \left[i \sum_{j=1}^{M} \left\{ i \boldsymbol{z}^\dagger(j) \Delta \boldsymbol{z}(j) - \Delta t \boldsymbol{z}^\dagger(j) h \boldsymbol{z}(j-1) \right\} \right] . \end{aligned} \tag{5.21}$$

In this way the importance of the compactness in the domain of multipliers can be recognized, which convinces us that the use of projection operator P_k (3.68) given in section 3.2 is indispensable.

A Proof of the theorems

In this appendix, we prove our main theorems in section 3.1.

A.1 Theorem 3.2

The statement is

$$\int_{\mathrm{U}(n)} \frac{dg}{(\det g)^p} \exp\{\mathrm{tr}(gX)\} = \mathcal{N}(n,p) |X|^p, \tag{A.1}$$

with

$$\mathcal{N}(n,p) = \frac{0!1! \cdots (n-1)!}{p!(p+1)! \cdots (p+n-1)!} , \tag{A.2}$$

and the assumptions being given in the text.

We use the following facts without proof.

(I) Invariant measure on $\mathrm{U}(n)$:

$$dg \propto \frac{1}{(\det g)^n} \prod_{1 \leq i,j \leq n} dg_{ij} , \tag{A.3}$$

where dg_{ij}'s denote n^2 independent differentials. Each g_{ij} is complex and the number of independent components is n^2 in terms of real variables.

(II) Local decomposition of $\mathrm{U}(n)$:

$$g \in \mathrm{U}(n) \Rightarrow g = \begin{pmatrix} a & 0 \\ 0 & B \end{pmatrix} \exp \begin{pmatrix} 0 & -\alpha^\dagger \\ \alpha & 0 \end{pmatrix} , \tag{A.4}$$

where $a \in \mathrm{U}(1),\ B \in \mathrm{U}(n-1)$ and $\alpha \in \boldsymbol{C}^{n-1}$ is the parameter for $\boldsymbol{C}P^{n-1}$. Rewrite (A.4) to

$$\begin{pmatrix} a & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{1+|\xi|^2}} & -\frac{1}{\sqrt{1+|\xi|^2}}\xi^\dagger \\ \xi\frac{1}{\sqrt{1+|\xi|^2}} & \frac{1}{\sqrt{1_{n-1}+\xi\xi^\dagger}} \end{pmatrix}, \quad \xi = \frac{\alpha}{|\alpha|}\tan|\alpha| , \tag{A.5}$$

so that

$$dg \propto d\mu_{n-1}(\xi)\frac{da}{a}\frac{\prod dB_{ij}}{(\det B)^{n-1}} , \tag{A.6}$$

where the measure has been decomposed into $\boldsymbol{C}P^{n-1}$, $\mathrm{U}(1)$ and $\mathrm{U}(n-1)$ in that order. Therefore a repeated use of the procedure results in

$$dg \propto \prod_{j=1}^{n-1} d\mu_j(\xi^{(j)}) \prod_{i=1}^{n} \frac{da_i}{a_i} , \tag{A.7}$$

that is, the invariant measure of $\mathrm{U}(n)$ is given by the product of $\boldsymbol{C}P^j$'s measure($1 \leq j \leq n-1$) and the tori of $\mathrm{U}(n)$, which corresponds to the local decomposition of $\mathrm{U}(n)$:

$$\begin{aligned} &\frac{\mathrm{U}(n)}{\mathrm{U}(1)\times\mathrm{U}(n-1)} \times \frac{\mathrm{U}(n-1)}{\mathrm{U}(1)\times\mathrm{U}(n-2)} \times \cdots \times \frac{\mathrm{U}(2)}{\mathrm{U}(1)\times\mathrm{U}(1)} \times \mathrm{U}(1)^n \\ \cong\ & \boldsymbol{C}P^{n-1} \times \boldsymbol{C}P^{n-2} \times \cdots \boldsymbol{C}P^1 \times \mathrm{U}(1)^n . \end{aligned} \tag{A.8}$$

(III) Integration formula on $\boldsymbol{C}P^N$:

$$\frac{(k+N)!}{k!}\int \frac{(d\xi d\bar{\xi})^N}{\pi^N(1+|\xi|^2)^{N+1+k}}(1+a^\dagger\xi)^k(1+\xi^\dagger b)^k = (1+a^\dagger b)^k \tag{A.9}$$

holds for ${}^\forall a, b \in \boldsymbol{C}^N$ and ${}^\forall k \in \boldsymbol{Z}_+$.

Since both sides of (A.1) are regular functions of x_{ij}, the case, $|X|=0$, can be regarded as a limit of $|X| \neq 0$. Then X can be assumed without loss of generality as

$$X = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \alpha \in \boldsymbol{C}, \beta^T, \gamma \in \boldsymbol{C}^{n-1}, \delta \in M(n-1;\boldsymbol{C}) , \tag{A.10}$$

with $\alpha \det\delta \neq 0$.

The proof is done by induction:

[I] For $n=1$, (A.1) is verified by a direct calculation:

$$\oint \frac{da}{2\pi i a^{k+1}} \sum_{k=0}^{\infty} \frac{1}{k!}(aX)^k = \frac{X^k}{k!} \ . \tag{A.11}$$

[II] Assume (A.1) holds for $n \le m$. Then adopt (A.5) for $g \in \mathrm{U}(m+1)$ to find

$$\operatorname{tr} g \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = a \frac{\alpha - \xi^\dagger \gamma}{\sqrt{1+|\xi|^2}} + \operatorname{tr}\left\{ B \left(\frac{\xi\beta}{\sqrt{1+|\xi|^2}} + \frac{1}{\sqrt{1_m + \xi\xi^\dagger}}\delta \right) \right\} \ . \tag{A.12}$$

According to the assumption of induction, when $n = m+1$, the integration with respect to a and B in the left hand side of (A.1) gives

$$\begin{aligned} &\frac{\mathcal{N}(m,p)}{p!} \int \frac{m!(d\xi d\bar{\xi})^m}{\pi^m (1+|\xi|^2)^{m+1}} \left\{ \frac{\alpha - \xi^\dagger\gamma}{\sqrt{1+|\xi|^2}} \right\}^p \\ &\qquad \times \left\{ \det \left(\frac{\xi\beta}{\sqrt{1+|\xi|^2}} + \frac{1}{\sqrt{1_m + \xi\xi^\dagger}}\delta \right) \right\}^p \ . \end{aligned} \tag{A.13}$$

By means of a relation

$$\det \left(\frac{\xi\beta}{\sqrt{1+|\xi|^2}} + \frac{1}{\sqrt{1_m + \xi\xi^\dagger}}\delta \right) = \frac{1 + \beta\delta^{-1}\xi}{\sqrt{1+|\xi|^2}} \det\delta \ , \tag{A.14}$$

(A.13) is rewritten as

$$\frac{m!\mathcal{N}(m,p)}{p!} (\alpha \det\delta)^p \int \frac{(d\xi d\bar{\xi})^m}{\pi^m (1+|\xi|^2)^{m+1+p}} (1 - \xi^\dagger\gamma\alpha^{-1})^p (1 + \beta\delta^{-1}\xi)^p \ , \tag{A.15}$$

which finally turns out, with the aid of (A.9), to be

$$\begin{aligned} \text{(A.15)} &= \frac{m!}{(p+m)!}\mathcal{N}(m,p)(\alpha\det\delta)^p (1 - \beta\delta^{-1}\gamma\alpha^{-1})^p \\ &= \frac{m!}{(p+m)!}\mathcal{N}(m,p) \left\{ \det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right\}^p \ . \end{aligned} \tag{A.16}$$

Hence (A.1) holds for $n = m+1$ as well as for $n \le m$.

This completes the proof.

A.2 Theorem 3.3

Next consider the second theorem:

$$\begin{aligned} |\partial_X|\,|X|^p &= p(p+1)\cdots(p+n-1)\,|X|^{p-1} \ , \\ p &= 0, \pm 1, \pm 2, \dots \ . \end{aligned} \tag{A.17}$$

A.2.1 Proof for $p \geq 0$.

In this case, the formula (A.1) is utilized to rewrite the left hand side of (A.17) as

$$|\partial_X|\,|X|^p = \frac{1}{\mathcal{N}(n,1)\mathcal{N}(n,p)} \int_{\mathrm{U}(n)} \frac{dg_1 dg_2}{\det g_1 (\det g_2)^p} \exp\left\{\mathrm{tr}(g_1 \partial_X)\right\} \exp\left\{\mathrm{tr}(g_2 X)\right\}. \quad \text{(A.18)}$$

Regard $\partial_{ij} = \partial/\partial x_{ij}$ and x_{ij} as operators upon functions of x_{ij} so that the both sides of (A.18) are implied as acting on 1. Then use is made of the Campbell-Baker-Hausdorff formula in the right hand side to interchange two exponential factors:

$$\begin{aligned} \text{r.h.s. of (A.18)} &= \frac{1}{\mathcal{N}(n,1)\mathcal{N}(n,p)} \int_{\mathrm{U}(n)} \frac{dg_1 dg_2}{\det g_1 (\det g_2)^p} \\ &\quad \times \exp\left\{\mathrm{tr}(g_1 g_2)\right\} \exp\left\{\mathrm{tr}(g_2 X)\right\} \exp\left\{\mathrm{tr}(g_1 \partial_X)\right\}\ , \end{aligned} \quad \text{(A.19)}$$

so that the last exponential factor can be dropped. Finally g_1 integration leads to

$$\begin{aligned} |\partial_X|\,|X|^p &= \frac{1}{\mathcal{N}(n,p)} \int_{\mathrm{U}(n)} \frac{dg}{(\det g)^{p-1}} \exp\left\{\mathrm{tr}(gX)\right\} \\ &= \frac{\mathcal{N}(n,p-1)}{\mathcal{N}(n,p)} |X|^{p-1} \\ &= p(p+1)\cdots(p+n-1)|X|^{p-1}, \end{aligned} \quad \text{(A.20)}$$

which complete the proof for $p \in \boldsymbol{Z}_+$. Note that there is no restriction to $X \in M(n;\boldsymbol{C})$ in this case.

A.2.2 Proof for $p < 0$.

In this case, note the following relation:

$$\begin{aligned} |\partial_X|\,|X|^{-p} &= |\partial_X| \int \left(\frac{dz d\bar{z}}{\pi}\right)^{np} \exp\left\{-\mathrm{tr}(XZZ^\dagger)\right\} \\ &= (-1)^n \int \left(\frac{dz d\bar{z}}{\pi}\right)^{np} |ZZ^\dagger| \exp\left\{-\mathrm{tr}(XZZ^\dagger)\right\}\ , \end{aligned} \quad \text{(A.21)}$$

for

$$X = A + iB, \quad A^\dagger = A, \quad B^\dagger = B, \quad A > 0\ , \quad \text{(A.22)}$$

where we have put $p \to -p$ so that p is positive here and hereafter.

First consider the case $p \leq n-1$. If we notice a relation

$$\det ZZ^\dagger = \det \begin{pmatrix} Z & 0 \end{pmatrix} \begin{pmatrix} Z^\dagger \\ 0 \end{pmatrix} = 0\ , \quad \text{(A.23)}$$

we find that the right hand side of (A.21) trivially vanishes when $p = 0, \ldots, n-1$. Hence it is enough to examine the case $p \geq n$. Before proceeding, we recall a well known fact: under the condition (A.22) A and B are simultaneously diagonalized by means of an appropriate invertible matrix K such that

$$K^\dagger A K = 1_n, \quad K^\dagger B K = B_D = \mathrm{diag}(\beta_1, \ldots, \beta_n) \ . \tag{A.24}$$

Accordingly a change of variables, $Z \mapsto Z' = K^{-1}Z$, gives

$$\text{(A.21)} = \frac{(-1)^n}{|A|^{p+1}} \int \left(\frac{dz d\bar{z}}{\pi}\right)^{np} |ZZ^\dagger| \exp\Big[-\operatorname{tr}\{(1_n + iB_D)ZZ^\dagger\}\Big] . \tag{A.25}$$

Then rewrite the matrix $1_n + iB_D$ as

$$1_n + iB_D = \begin{pmatrix} 1+i\beta_1 & & 0 \\ & \ddots & \\ 0 & & 1+i\beta_n \end{pmatrix} = F\Phi, \tag{A.26}$$

$$F \equiv \begin{pmatrix} f_1 & & 0 \\ & \ddots & \\ 0 & & f_n \end{pmatrix}, \quad \Phi \equiv \begin{pmatrix} e^{i\phi_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\phi_n} \end{pmatrix}, \tag{A.27}$$

$$0 < f_i, \qquad -\frac{\pi}{2} < \phi_i < \frac{\pi}{2} \quad (i = 1, \ldots, n) \ . \tag{A.28}$$

Further a change of variables $Z \mapsto Z' = \sqrt{F}Z$ leads to

$$\text{(A.25)} = \frac{(-1)^n}{|A|^{p+1}|F|^{p+1}} \int \left(\frac{dz d\bar{z}}{\pi}\right)^{np} |ZZ^\dagger| \exp\Big\{-\operatorname{tr}(\Phi ZZ^\dagger)\Big\} \ , \tag{A.29}$$

which is rewritten, by use of the formula (A.1), as

$$\text{(A.29)} = \frac{(-1)^n n!}{|A|^{p+1}|F|^{p+1}} \lim_{\varepsilon \to 0} \int_{\mathrm{U}(n)} \frac{dg}{\det g} \int \left(\frac{dz d\bar{z}}{\pi}\right)^{np} \exp\Big[-\operatorname{tr}\{(\Phi - ge^{-\varepsilon})ZZ^\dagger\}\Big] \ . \tag{A.30}$$

Now the Gaussian integration with respect to Z is performed to be

$$\text{(A.30)} = \frac{(-1)^n n!}{|A|^{p+1}|F|^{p+1}} \lim_{\varepsilon \to 0} \int_{\mathrm{U}(n)} \frac{dg}{\det g} \frac{1}{\det(\Phi - ge^{-\varepsilon})^p} \ , \tag{A.31}$$

which becomes after a change of variable $g \mapsto \Phi^{-1}g$ to

$$\begin{aligned} \text{(A.31)} \quad &= \quad \frac{(-1)^n n!}{|A|^{p+1}|F|^{p+1}|\Phi|^{p+1}} \lim_{\varepsilon \to 0} \int_{\mathrm{U}(n)} \frac{dg}{\det g} \frac{1}{\det(1_n - ge^{-\varepsilon})^p} \\ &= \quad \frac{(-1)^n n!}{|X|^{p+1}} \lim_{\varepsilon \to 0} \int_{\mathrm{U}(n)} \frac{dg}{\det g} \frac{1}{\det(1_n - ge^{-\varepsilon})^p} \ . \end{aligned} \tag{A.32}$$

In view of (A.32) and (A.17) (with $p \to -p$), our remaining task is therefore to prove

$$\lim_{\varepsilon\to 0}\int_{\mathrm{U}(n)} \frac{dg}{\det g}\frac{1}{\det(1_n - ge^{-\varepsilon})^p} = \binom{p}{n}\,. \tag{A.33}$$

In order to perform the g-integration, recall the decomposition[27, 24],

$$g = \Omega g_0 \Omega^\dagger,\;\; g_0 = \mathrm{diag}(e^{i\theta_1},\ldots,e^{i\theta_n}),\;\; \Omega \in \mathrm{SU}(n), \tag{A.34}$$

and integrate Ω to obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\mathrm{U}(n)} \frac{dg}{\det g}\frac{1}{\det(1_n - ge^{-\varepsilon})^p}\\
=\;&\lim_{\varepsilon\to 0}\int_0^{2\pi}\left(\frac{d\theta}{2\pi}\right)^n \frac{1}{n!}\prod_{a<b}|e^{i\theta_a}-e^{i\theta_b}|^2\prod_{a=1}^{n}\frac{e^{-i\theta_a}}{(1-e^{-\varepsilon+i\theta_a})^p}\\
=\;&\lim_{\varepsilon\to 0}\int_0^{2\pi}\left(\frac{d\theta}{2\pi}\right)^n \frac{1}{n!}\sum_{\sigma,\tau\,\in\,\mathcal{S}_n}\mathrm{sgn}(\sigma\tau)\prod_{a=1}^{n}\sum_{l_a=0}^{\infty}\binom{p+l_a-1}{l_a}\\
&\qquad\qquad\times\exp\left[i\left\{l_a-1+\sigma(a)-\tau(a)\right\}\theta_a - l_a\varepsilon\right]\,,
\end{aligned} \tag{A.35}$$

which then yields after the θ's integrations and putting $\varepsilon\to 0$ to

$$(\mathrm{A.35}) = \begin{vmatrix}
p & \binom{p+1}{2} & \binom{p+2}{3} & \cdots & \binom{p+n-1}{n}\\
1 & p & \binom{p+1}{2} & \ddots & \vdots\\
0 & 1 & p & \ddots & \binom{p+2}{3}\\
\vdots & \ddots & \ddots & \ddots & \binom{p+1}{2}\\
0 & \cdots & 0 & 1 & p
\end{vmatrix}\,. \tag{A.36}$$

We denote this determinant by $\mathcal{D}(n,p),\; n=0,1,2,\ldots$ with defining $\mathcal{D}(0,p)=1$. Recall that our goal is now to show

$$\mathcal{D}(n,p) = \binom{p}{n} \tag{A.37}$$

Let us prove this again by induction: first notice the recursion relation, obtained by an expansion in the first row of (A.36),

$$\mathcal{D}(n,p) = \sum_{r=1}^{n}(-1)^{1+r}\binom{p+r-1}{r}\mathcal{D}(n-r,p)\,. \tag{A.38}$$

Assume (A.37) for $0\le m\le n-1$, then (A.38) reads

$$\begin{aligned}
\mathcal{D}(n,p) &= \sum_{r=1}^{n}(-1)^{1+r}\binom{p+r-1}{r}\binom{p}{n-r}\\
&= \frac{p}{n}\sum_{r=1}^{n}(-1)^{1+r}\binom{n}{r}\binom{p+r-1}{p+r-n}\,.
\end{aligned} \tag{A.39}$$

Utilizing the generating function

$$\binom{p+r-1}{p+r-n} = \frac{1}{p!}\left(\frac{d}{dx}\right)^p\Big|_{x=0}\sum_{l=0}^{\infty}\binom{l+n-1}{l}x^{l+n-r} \ , \tag{A.40}$$

we find

$$\begin{aligned}
&\binom{p}{n} - \mathcal{D}(n,p) = \frac{p}{n}\sum_{r=0}^{n}(-1)^r\binom{n}{r}\binom{p+r-1}{p+r-n} \\
= \ & \frac{p}{n}\frac{1}{p!}\left(\frac{d}{dx}\right)^p\Big|_{x=0}\sum_{r=0}^{n}(-1)^r\binom{n}{r}\sum_{l=0}^{\infty}\binom{l+n-1}{l}x^{l+n-r} \\
= \ & \frac{p}{n}\frac{1}{p!}\left(\frac{d}{dx}\right)^p\Big|_{x=0}\sum_{r=0}^{n}\binom{n}{r}(-1)^r x^{n-r}\left(\frac{1}{1-x}\right)^n \\
= \ & (-1)^n\frac{p}{n}\frac{1}{p!}\left(\frac{d}{dx}\right)^p\Big|_{x=0} 1 \\
= \ & 0 \ .
\end{aligned} \tag{A.41}$$

Thus we have found that (A.37) is also valid for $m = n$. This completes the proof.

B Feynman kernel and the WKB approximation

In order to make a clear connection to the D-H theorem, we have concentrated on the character formula in the text. However, from the quantum mechanical point of view, the Feynman kernel is regarded primitive so that in this appendix a brief sketch is presented to show the way to a path integral representation and discuss the WKB approximation.

B.1 Derivation of the Feynman kernel (method 1)

Take a Hermitian matrix,

$$H \equiv \begin{pmatrix} A & B \\ B^\dagger & D \end{pmatrix} \in \mathrm{H}(N) \tag{B.1}$$

with the same convention given in (2.14) then consider the Feynman kernel

$$K_k(\xi_F, \xi_I; T) \equiv \langle \xi_F; k|\rho_k(e^{-iHT})|\xi_I; k\rangle \ . \tag{B.2}$$

Follow a similar procedure from (4.2) to (4.6) to obtain

$$\begin{aligned}
&K_k(\xi_F, \xi_I; T) \\
= \ & \frac{1}{[\det\{(1_n + \xi_F^\dagger\xi_F)(1_n + \xi_I^\dagger\xi_I)\}]^{k/2}}\lim_{M\to\infty}\int\prod_{i=1}^{M-1}d\mu(\xi(i); k)
\end{aligned}$$

$$\times \exp\left[-k\sum_{j=1}^{M-1}\mathrm{tr}\log(1_n+\xi^\dagger(j)\xi(j))+k\sum_{j=1}^{M}\mathrm{tr}\log(1_n+\xi^\dagger(j)\xi(j-1))\right]$$
$$\times \exp\left[-ik\Delta t\sum_{j=1}^{M}\mathrm{tr}\{P(\xi(j),\xi(j-1))H\}\right]\ , \tag{B.3}$$

where $\xi(0)=\xi_I,\ \xi(M)=\xi_F$. Introduce a one-parameter subgroup of $\mathrm{U}(N)$

$$g(t)=\exp(-iHt)=\begin{pmatrix}\alpha(t) & \beta(t)\\ \gamma(t) & \delta(t)\end{pmatrix}\ ,\quad t\in\boldsymbol{R}, \tag{B.4}$$

and write with the abbreviation $\alpha(j\Delta t)=\alpha(j)$ etc.

$$\mathbf{L}(i,j)\equiv\alpha(i-j)+\xi^\dagger(i)\gamma(i-j)+\beta(i-j)\xi(j)+\xi^\dagger(i)\delta(i-j)\xi(j)\ , \tag{B.5}$$

then discard $O((\Delta t)^2)$ terms to obtain

$$\begin{aligned}&K_k(\xi_F,\xi_I;T)\\ =\ &\frac{1}{[\det\{(1_n+\xi_F^\dagger\xi_F)(1_n+\xi_I^\dagger\xi_I)\}]^{k/2}}\lim_{M\to\infty}\int\prod_{i=1}^{M-1}d\mu(\xi(i);k)\\ &\times\exp\left[k\sum_{j=1}^{M}\mathrm{tr}\log\mathbf{L}(j,j-1)-k\sum_{j=1}^{M-1}\mathrm{tr}\log\mathbf{L}(j,j)\right]\ ,\end{aligned} \tag{B.6}$$

whose j-th integration part is

$$\int\frac{d\mu(\xi(j);k)}{\{\det(1_n+\xi^\dagger(j)\xi(j))\}^k}\left[\det\left\{\mathbf{L}(j+1,j)\mathbf{L}(j,j-1)\right\}\right]^k\ . \tag{B.7}$$

To carry out this integration, write

$$\begin{aligned}&\det\{\mathbf{L}(j+1,j)\mathbf{L}(j,j-1)\}\\ =\ &\det\{\alpha(1)+\xi^\dagger(j+1)\gamma(1)\}\det\left\{1_n+\{\xi^\dagger(j+1)*g(1)\}\xi(j)\right\}\\ &\times\det\left\{1_n+\xi^\dagger(j)\{g(1)*\xi(j-1)\}\right\}\det\{\alpha(1)+\beta(1)\xi(j-1)\}\ ,\end{aligned} \tag{B.8}$$

where

$$\begin{aligned}g(j)*\xi\ &\equiv\ \{\gamma(j)+\delta(j)\xi\}\{\alpha(j)+\beta(j)\xi\}^{-1}\ ,\\ \xi^\dagger*g(j)\ &\equiv\ \{\alpha(j)+\xi^\dagger\gamma(j)\}^{-1}\{\beta(j)+\xi^\dagger\delta(j)\}\ .\end{aligned} \tag{B.9}$$

Utilizing the formula (3.48) we find

$$\begin{aligned}(\mathrm{B}.7)\ =\ &\det\{\alpha(1)+\xi^\dagger(j+1)\gamma(1)\}^k\\ &\times\left[\det\left\{1_n+\{\xi^\dagger(j+1)*g(1)\}\{g(1)*\xi(j-1)\}\right\}\right]^k\\ &\times\det\{\alpha(1)+\beta(1)\xi(j-1)\}^k\ ,\end{aligned} \tag{B.10}$$

which is nothing but

$$(\mathrm{B.7}) = \{\det \mathbf{L}(j+1, j-1)\}^k \ , \tag{B.11}$$

since $g(t)$ is an element of the one-parameter subgroup (B.4). Hence after $M-1$ times of this manipulation we obtain

$$K_k(\xi_F, \xi_I; T) = \left[\frac{\det\{\alpha(T) + \xi_F^\dagger \gamma(T) + \beta(T)\xi_I + \xi_F^\dagger \delta(T)\xi_I\}}{\det\{(1_n + \xi_F^\dagger \xi_F)(1_n + \xi_I^\dagger \xi_I)\}^{1/2}}\right]^k \ . \tag{B.12}$$

B.2 Derivation of the Feynman kernel (method 2)

Alternative representation can be given by the Schwinger boson technique. First introduce an integral representation of the inner product between coherent states such that

$$\begin{aligned}\langle \xi; k|\eta; k\rangle \ &= \ \frac{\mathcal{N}(n, N-n+k)}{\mathcal{N}(n,k)} \int \left(\frac{d\zeta d\bar{\zeta}}{\pi}\right)^{n^2} \{\det(\zeta^\dagger\zeta)\}^{N-n} \\ &\quad \times \int_{\mathrm{U}(n)} \frac{dg}{(\det g)^k} \langle Z(\xi,\zeta)|Z(\eta,\zeta)g\rangle \ ,\end{aligned} \tag{B.13}$$

where

$$|Z(\xi,\zeta)\rangle \equiv \exp\left\{\mathrm{tr}(a^\dagger Z(\xi,\zeta) - Z^\dagger(\xi,\zeta)a)\right\}|0\rangle \tag{B.14}$$

is the canonical coherent state with $Z(\xi,\zeta)$ being defined by

$$\begin{aligned}&\xi \in M(N-n, n; \boldsymbol{C}), \ \ \zeta \in M(n; \boldsymbol{C}) \\ \mapsto \ \ & Z(\xi,\zeta) \equiv \begin{pmatrix} 1_n \\ \xi \end{pmatrix} \frac{1}{\sqrt{1_n + \xi^\dagger\xi}}\zeta \ \in M(N, n; \boldsymbol{C}) \ .\end{aligned} \tag{B.15}$$

(The relation (B.13) can be verified by use of the formulae 3.2 and 3.3.) Then the Feynman kernel (B.2) is expressed as

$$\begin{aligned}& K_k(\xi_F, \xi_I; T) \\ = \ & \frac{\mathcal{N}(n, N-n+k)}{\mathcal{N}(n,k)} \int \left(\frac{d\zeta d\bar{\zeta}}{\pi}\right)^{n^2} \{\det(\zeta^\dagger\zeta)\}^{N-n} \\ & \times \int_{\mathrm{U}(n)} \frac{dg}{(\det g)^k} \langle Z(\xi_F,\zeta)|\exp(-i\hat{H}T)|Z(\xi_I,\zeta)g\rangle \ ,\end{aligned} \tag{B.16}$$

where $\hat{H}$ has been given by (4.25) for H in (B.1). The last quantity in (B.16) is given by

$$\begin{aligned}& \langle Z(\xi_F,\zeta)|\exp(-i\hat{H}T)|Z(\xi_I,\zeta)g\rangle \\ = \ \ & \exp\left\{-\mathrm{tr}(\zeta^\dagger\zeta)\right\} \lim_{M\to\infty} \int \prod_{i=1}^{M-1} \left(\frac{dz(i)d\bar{z}(i)}{\pi}\right)^{Nn} \\ & \times \exp\left[-\mathrm{tr}\left\{\sum_{j=1}^{M-1} Z^\dagger(j)Z(j) - \sum_{j=1}^{M} Z^\dagger(j)g(1)Z(j-1)\right\}\right] \ ,\end{aligned} \tag{B.17}$$

with $Z(0) = Z(\xi_I, \zeta)g,\ Z^\dagger(M) = Z^\dagger(\xi_F, \zeta)$, which becomes

$$\text{(B.17)} \ = \exp\left[-\operatorname{tr}\left\{\zeta^\dagger\zeta - g\zeta^\dagger\mathbf{K}(\xi_F, \xi_I; T)\zeta\right\}\right] \ , \tag{B.18}$$

where

$$\begin{aligned}
& \mathbf{K}(\xi_F, \xi_I; T) \\
\equiv\ & (1_n + \xi_F^\dagger\xi_F)^{-1/2}\mathbf{L}(M, 0)(1_n + \xi_I^\dagger\xi_I)^{-1/2} \\
=\ & (1_n + \xi_F^\dagger\xi_F)^{-1/2}\{\alpha(T) + \xi_F^\dagger\gamma(T) + \beta(T)\xi_I + \xi_F^\dagger\delta(T)\xi_I\}(1_n + \xi_I^\dagger\xi_I)^{-1/2} \ .
\end{aligned} \tag{B.19}$$

Substituting (B.18) into (B.16) and carrying out integrations with respect to g and ζ, we find

$$\begin{aligned}
K_k(\xi_F, \xi_I; T) &= \{\det\mathbf{K}(\xi_F, \xi_I; T)\}^k \\
&= \left[\frac{\det\{\alpha(T) + \xi_F^\dagger\gamma(T) + \beta(T)\xi_I + \xi_F^\dagger\delta(T)\xi_I\}}{\det\{(1_n + \xi_F^\dagger\xi_F)(1_n + \xi_I^\dagger\xi_I)\}^{1/2}}\right]^k
\end{aligned} \tag{B.20}$$

which of course matches to the result obtained in the previous section, however, this rewrite can again provide an interpretation of the WKB exactness.

B.3 The WKB approximation

From the path integral expression (B.6), the action is given by

$$S = \sum_{j=1}^{M}\operatorname{tr}\log\mathbf{L}(j, j-1) - \sum_{j=1}^{M-1}\operatorname{tr}\log\mathbf{L}(j, j) \ , \tag{B.21}$$

and therefore equations of motion read

$$\{\gamma(1) + \delta(1)\xi(j-1)\}\mathbf{L}^{-1}(j, j-1) \ = \ \xi(j)\mathbf{L}^{-1}(j, j) \ , \tag{B.22}$$

$$\mathbf{L}^{-1}(j+1, j)\{\beta(1) + \xi^\dagger(j+1)\delta(1)\} \ = \ \mathbf{L}^{-1}(j, j)\xi^\dagger(j) \ , \tag{B.23}$$

for $1 \le j \le M-1$. These can be solved locally by

$$\xi(j) \ = \ \{\gamma(1) + \delta(1)\xi(j-1)\}\{\alpha(1) + \beta(1)\xi(j-1)\}^{-1} \ , \tag{B.24}$$

$$\xi^\dagger(j) \ = \ \{\alpha(1) + \xi^\dagger(j+1)\gamma(1)\}^{-1}\{\beta(1) + \xi^\dagger(j+1)\delta(1)\} \ , \tag{B.25}$$

so that the classical solution is, after taking account of the boundary condition $\xi(0) = \xi_I; \xi(M) = \xi_F$, found such that

$$\xi_c(j) \ = \ \{\gamma(j) + \delta(j)\xi_I\}\{\alpha(j) + \beta(j)\xi_I\}^{-1} \ , \tag{B.26}$$

$$\xi_c^\dagger(j) \ = \ \{\alpha(M-j) + \xi_F^\dagger\gamma(M-j)\}^{-1}\{\beta(M-j) + \xi_F^\dagger\delta(M-j)\} \ . \tag{B.27}$$

Here two comments are in order: the first is on the fact that $\xi_c(j)$ and $\xi_c^\dagger(j)$ are not Hermitian conjugated each other. This is not so surprising, although it is often emphasized. A similar situation can be met even in a simple Gaussian integration:

$$\int \frac{dzd\bar{z}}{\pi} \exp(-\bar{z}z + \bar{a}z + \bar{z}b) \ , \tag{B.28}$$

for $a, b \in \boldsymbol{C}$. Complete the square to find

$$\bar{z}z - \bar{a}z - \bar{z}b = (\bar{z} - \bar{a})(z - b) - \bar{a}b \ , \tag{B.29}$$

then shift the variable $z(\bar{z})$ by the amount of $b(\bar{a})$ to obtain

$$\int \frac{dzd\bar{z}}{\pi} \exp(-\bar{z}z + \bar{a}b) = e^{\bar{a}b} \ . \tag{B.30}$$

The shift of z and $\bar{z}$ is asymmetric but there is no problem. The point is that they should be regarded independent. In our case we have two independent sources ξ_I and $\xi_F^\dagger$ which are not always Hermitian conjugated each other.

The second remark is on the fact that there is no overspecification problem which is also often stressed by authors in the frame work of continuum version of path integral; since it seems two boundary conditions in the first order differential equation. But under the discrete time formalism there is no room for appearance of $\xi_I^\dagger$ and ξ_F as to the boundary conditions.

Now turn back to the main subject: we must evaluate the action (B.21) and its Hessian at classical solutions (B.26) and (B.27). To this end note the following relation:

$$\mathbf{L}(j,j)_c\{\alpha(j) + \beta(j)\xi_I\} = \mathbf{L}(j,j-1)_c\{\alpha(j-1) + \beta(j-1)\xi_I\} \ , \tag{B.31}$$

where the subscript c designates quantities of the classical solutions. By use of this relation, we can easily evaluate the classical action:

$$\begin{aligned} S_c &\equiv \sum_{j=1}^{M} \operatorname{tr}\log \mathbf{L}(j,j-1)_c - \sum_{j=1}^{M-1} \operatorname{tr}\log \mathbf{L}(j,j)_c \\ &= \operatorname{tr}\log\left[\mathbf{L}(M,M-1)_c\{\alpha(M-1) + \beta(M-1)\xi_I\}\right] \\ &= \operatorname{tr}\log \mathbf{L}(M,0) \ , \end{aligned} \tag{B.32}$$

with $\mathbf{L}(M,0) \in M(n;\boldsymbol{C})$ being given by (B.5): explicitly $\mathbf{L}(M,0) = \alpha(T) + \xi_F^\dagger\gamma(T) + \beta(T)\xi_I + \xi_F^\dagger\delta(T)\xi_I$. Let us introduce another one-parameter subgroup by

$$\tilde{g}(t) \equiv \begin{pmatrix} \delta^\dagger(t) & -\beta^\dagger(t) \\ -\gamma^\dagger(t) & \alpha^\dagger(t) \end{pmatrix} \ , \tag{B.33}$$

to define

$$\tilde{\mathbf{L}}(i,j) \equiv \delta^\dagger(i-j) - \xi(j)\gamma^\dagger(i-j) - \beta^\dagger(i-j)\xi^\dagger(i) + \xi(j)\alpha^\dagger(i-j)\xi^\dagger(i) \ . \tag{B.34}$$

Then the Hessian of the action is found as

$$\frac{\partial^2 S}{\partial \xi_{ia}(l)\partial \bar{\xi}_{jb}(m)} = -\delta_{l,m}\tilde{\mathbf{L}}^{-1}(m,m)_{ji}\mathbf{L}^{-1}(m,m)_{ab} + \delta_{l,m-1}\tilde{\mathbf{L}}^{-1}(m,m-1)_{ji}\mathbf{L}^{-1}(m,m-1)_{ab} \ . \tag{B.35}$$

Shifting the variables

$$\xi(j) = \xi_c(j) + z(j),\ \ \xi^\dagger(j) = \xi_c^\dagger(j) + z^\dagger(j)\ , \tag{B.36}$$

we approximate the action up to the bilinear terms of z and $z^\dagger$:

$$\begin{aligned} S \ \ &\sim \ \ S_c - \sum_{j=1}^{M-1} \mathrm{tr}\left\{z^\dagger(j)\tilde{\mathbf{L}}^{-1}(j,j)_c z(j)\mathbf{L}^{-1}(j,j)_c\right\} \\ &\quad + \sum_{j=2}^{M-1} \mathrm{tr}\left\{z^\dagger(j)\tilde{\mathbf{L}}^{-1}(j,j-1)_c z(j-1)\mathbf{L}^{-1}(j,j-1)_c\right\}\ , \end{aligned} \tag{B.37}$$

The measure is also approximated as

$$d\mu(\xi(j);k)\bigg|_c \overset{k\to\infty}{\sim} \frac{k^{n(N-n)}}{\{\det \mathbf{L}(j,j)_c\}^N}\left(\frac{dz(j)d\bar{z}(j)}{\pi}\right)^{n(N-n)}\left\{1+O(k^{-1})\right\}\ , \tag{B.38}$$

where use has been made of the definition of the measure (3.42) with (4.14). Therefore in view of (B.37) and (B.38) the Gaussian integration of $\bar{z}(j), z(j)$ gives

$$\det\left\{\tilde{\mathbf{L}}^{-1}(j,j)_c \otimes (\mathbf{L}^{-1}(j,j)_c)^{\mathcal{T}}\right\}^{-1} = \{\det \mathbf{L}(j,j)_c\}^N \tag{B.39}$$

which together with $(\pi/k)^{n(N-n)}$ cancels the prefactor in (B.38), yielding the leading contribution to the WKB approximation:

$$\begin{aligned} K_k(\xi_F,\xi_I;T) \ \ &\overset{k\to\infty}{\sim} \ \ \frac{1}{[\det\{(1_n+\xi_F^\dagger\xi_F)(1_n+\xi_I^\dagger\xi_I)\}]^{k/2}}\exp\{k\,\mathrm{tr}\log\mathbf{L}(M,0)\} \\ &= \ \ \left[\frac{\det\{\alpha(T)+\xi_F^\dagger\gamma(T)+\beta(T)\xi_I+\xi_F^\dagger\delta(T)\xi_I\}}{\det\{(1_n+\xi_F^\dagger\xi_F)(1_n+\xi_I^\dagger\xi_I)\}^{1/2}}\right]^k\ . \end{aligned} \tag{B.40}$$

The result is equal to the full form of the Feynman kernel (B.12), (B.20). Thus *the WKB approximation is again exact for the Feynman kernel*, which can also be clarified in terms of (B.16) as well as (B.17); since the integrand is Gaussian.

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